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# How to find semimartingale decompositions relative to enlarged filtrations

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Let $S$ be a semimartingale relative to $(\mathcal{F}_t)$

$$\mathcal{G}_t \supset \mathcal{F}_t \qquad \text{enlargement}$$

Questions:

- Is $S$ a $(\mathcal{G}_t)$-semimartingale also?
- If yes, how do the new semimartingale decompositions look like?
- $(H \cdot_{\mathcal{F}} S)$ defined $\Longrightarrow$ $(H \cdot_{\mathcal{G}} S)$ defined also?

## Application to mathematical finance

Financial markets with insiders

intrinsic perspective of the price for insider or normal investor

$$S_t = M_t + \int_0^t \alpha_s d\langle M, M\rangle_s$$

- optimal investment strategy

$$\theta_t^* = x\alpha_t \mathcal{E}(\alpha \cdot M)_t$$

- maximal expected logarithmic utility

$$u(x) = \log(x) + \frac{1}{2} E \int_0^T \alpha_s^2 d\langle M, M\rangle_s$$

$\Longrightarrow$ investment & utility depend on the semimartingale dec.

# Semimartingale decompositions:

Let $M$ be an $(\mathcal{F}_t)$-martingale

How to find a $(\mathcal{G}_t)$-decomposition $M = N + A$ ?

**Definition 1** *A $(\mathcal{G}_t)$-predictable process $\mu$ such that*

$$M - \int_0^{\cdot} \mu_t \; d\langle M, M\rangle_t \quad \textit{is a} \quad (\mathcal{G}_t) - \textit{local martingale}$$

*is called information drift of $(\mathcal{G}_t)$ with respect to $M$.*

# Initial enlargements

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L) \qquad (L \text{ random variable})$$

probability conditioned on the new information

$$P(\cdot|L)$$

**1. Observation:** The enlargement $(\mathcal{F}_t) \to (\mathcal{G}_t)$ corresponds to the random change of probability $P \to P(\cdot|L)$

If there are no singularities

$$P(\cdot|L) \ll P \text{ on } \mathcal{F}_t \text{ for all } t \qquad (\text{ Jacod's condition}),$$

then

$$M \quad \text{martingale rel. to } (\mathcal{F}_t) \Longrightarrow M \quad \text{semimartingale rel. to } P(\cdot|L)$$

**2. Observation** Girsanov's theorem $\Longrightarrow$ semimartingale decompositions

For all $x$ let

$$p_t^x(\omega) := \frac{P(d\omega|L=x)}{P(d\omega)}\Big|_{\mathcal{F}_t}$$

be the conditional density.

$$\begin{aligned}
& M && (\mathcal{F}_t, P) - \text{martingale} \\
\Longrightarrow\quad & M - \frac{1}{p^x}\cdot\langle M, p^x\rangle && (\mathcal{F}_t, P(\cdot|L=x)) - \text{martingale} \\
\Longrightarrow\quad & M - \frac{1}{p^x}\cdot\langle M, p^x\rangle && (\mathcal{G}_t, P(\cdot|L=x)) - \text{martingale} \\
\Longrightarrow\quad & M - \frac{1}{p^L}\cdot\langle M, p^L\rangle && (\mathcal{G}_t, P) - \text{martingale}
\end{aligned}$$

$$M - \frac{1}{p^L} \cdot \langle M, p^L \rangle \qquad (\mathcal{G}_t, P) - \text{martingale}$$

**Theorem 1** *If*

$$p_t^x = p_0^x + \int_0^t \alpha_s^x dM_s + \ ortho.\ martingale$$

*then*

$$M_t - \int_0^t \frac{\alpha_s^{L(\omega)}}{p_s^{L(\omega)}} d\langle M, M \rangle_s \quad \textit{is a} \quad (\mathcal{G}_t) - \textit{local martingale.}$$

**Remarks:**

1) inf. drift $= \frac{\alpha_s^{L(\omega)}}{p_s^{L(\omega)}} =$ variational derivative of the logarithm of $p^L$

2) All we need is:

$$\alpha^x(s)\ P_L(dx) \quad \ll \quad p_s^x\ P_L(dx)\ =\ P(L \in dx | \mathcal{F}_s)$$

## Information drift via Malliavin calculus

On the Wiener space:

information drift = logarithmic Malliavin trace of the conditional probability relative to the new information

**Theorem 2** *(Imkeller, Pontier, Weisz 2000)*
*If*

$$D_t P(L \in dx | \mathcal{F}_t) \quad \ll \quad P(L \in dx | \mathcal{F}_t)$$

*then the* $(\mathcal{G}_t)$*-information drift is given by*

$$\frac{D_t p_t^{L(\omega)}(\omega)}{p_t^{L(\omega)}(\omega)}.$$

# General enlargements (continuous case)

Arbitrary enlargement $(\mathcal{G}_t) \supset (\mathcal{F}_t)$

**Aim:** General representation of the information drift of a continuous martingale $M$ wrt $(\mathcal{G}_t)$

**Assumption:** There exist $(\mathcal{F}_t^0)$ and $(\mathcal{G}_t^0)$ countably generated s. th. $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ are the smallest extensions with the usual conditions.

$\Longrightarrow$ reg. conditional probability $P_t(\omega, A)$ relative to $\mathcal{F}_t^0$ exists

Martingale property $\Longrightarrow$

$$P_t(\cdot, A) = P(A) + \int_0^t k_s(\cdot, A) dM_s + L_t^A,$$

where $\langle L^A, M \rangle = 0$.

**Condition (Abs):** $k_t(\omega,\cdot)$ is a signed measure on $\mathcal{G}^0_{t-}$ and satisfies

$$k_t(\omega,\cdot)\Big|_{\mathcal{G}^0_{t-}} \ll P_t(\omega,\cdot)\Big|_{\mathcal{G}^0_{t-}}$$

for $d\langle M,M\rangle \otimes P$-a.a $(\omega,t)$.

**Lemma 1** *There exists an $(\mathcal{F}_t \otimes \mathcal{G}_t)-$predictable process $\gamma$ such that for $d\langle M,M\rangle \otimes P$-a.a. $(\omega,t)$*

$$\gamma_t(\omega,\omega') = \frac{k_t(\omega,d\omega')}{P_t(\omega,d\omega')}\Big|_{\mathcal{G}^0_{t-}}.$$

**Theorem 3** *(A., Dereich, Imkeller 2005) The information drift of $M$ relative to $(\mathcal{G}_t)$ is given by*

$$\alpha_t(\omega) = \gamma_t(\omega,\omega)$$

**Question:** When is (Abs) satisfied?
How *strong* is the assumption (Abs)?

**Theorem 4**

There exists a square-integrable information drift $\Longrightarrow$ (Abs)

**Proof:** requires that $\sigma$-fields are countably generated

**Questions:** 1. Practical relevance?
2. What about martingales with jumps?

# Purely discontinuous martingales

$$X_t = \int_0^t \int_{\mathbb{R}_0} \psi(s,z)\ [\mu - \pi](ds, dz)$$

$\mu$ = Poisson random measure with compensator $\pi$

$\psi$ predictable and integrable

**Predictable representation property**

If $M$ square integrable $(\mathcal{F}_t)$-martingale, then there exists a predictable $\varphi \in L^2(\pi \otimes P)$ such that

$$M_t = M_0 + \int_0^t \int_{\mathbb{R}_0} \varphi(s,z)\ [\mu - \pi](ds, dz).$$

Arbitrary enlargement $(\mathcal{G}_t) \supset (\mathcal{F}_t)$

Conditional new information

$$P_t(\cdot, A) = P(A) + \int_0^t \int_{\mathbb{R}_0} k_s(z, A)[\mu - \pi](ds, dz).$$

$\nu$ = Levy measure

**Condition (Abs):** $\int_{\mathbb{R}_0} \psi_t(\omega, z) k_t(\omega, z, \cdot) d\nu(z)$ is a signed measure on $\mathcal{G}^0_{t-}$ and satisfies

$$\int_{\mathbb{R}_0} \psi_t(\omega, z) k_t(\omega, z, \cdot) d\nu(z)\Big|_{\mathcal{G}^0_{t-}} \ll P_t(\omega, \cdot)\Big|_{\mathcal{G}^0_{t-}},$$

for $P \otimes l$-a.a.$(\omega, t)$.

**Theorem 5** *There exists an* $(\mathcal{F}_t \otimes \mathcal{G}_t)-$*predictable* $\delta$ *such that for* $d\langle M, M\rangle \otimes P$*-a.a.* $(\omega, t)$

$$\delta_t(\omega, \omega') = \left.\frac{\int_{\mathbb{R}_0} \psi_t(\omega, z) k_t(\omega, z, d\omega') d\nu(z)}{P_t(\omega, d\omega')}\right|_{\mathcal{G}^0_{t-}}$$

*Moreover,*

$$\eta_t(\omega) = \delta_t(\omega, \omega)$$

*is the information drift of* $X$*, i.e.*

$$X_t - \int_0^t \eta_s \, ds \quad \textit{is a } (\mathcal{G}_t)\textit{-local martingale}$$

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## Calculating examples

General scheme:

- If $\mathcal{G}_t^0 = \mathcal{F}_t^0 \vee \mathcal{H}_t^0$, then it is enough to determine the density along $(\mathcal{H}_t^0)$, i.e.

$$\delta_t(\omega, \omega') = \left. \frac{\int_{\mathbb{R}_0} \psi_t(\omega, z) k_t(\omega, z, d\omega') d\nu(z)}{P_t(\omega, d\omega')} \right|_{\mathcal{H}_{t-}^0}.$$

- Determine the density by using a generalized Clark-Ocone formula:

$$k_t(\omega, z, A) = \text{ predictable projection of } D_{t,z} P_{t+}(\omega, A)$$

# A Clark-Ocone formula for Poisson random measures

Canonical space:

$\Omega$ = set of all integer valued signed measures $\omega$ on $[0,1] \times \mathbb{R} \setminus \{0\}$ s.th.

- $\omega(\{(t,z)\}) \in \{0,1\}$,
- $\omega(A \times B) < \infty$ if $\pi(A \times B) = \lambda(A)\nu(B) < \infty$.

random measure

$$\mu(\omega; A \times B) := \omega(A \times B)$$

$P$ = measure on $\Omega$ such that

$$\mu \text{ is a Poisson r.m. with compensator } \pi = \lambda \otimes \nu$$

# Picard's difference operator

## Definition:

$\epsilon^-_{(t,z)}$ and $\epsilon^+_{(t,z)} : \Omega \longrightarrow \Omega$ defined by

$$\epsilon^-_{(t,z)}\omega(A \times B) := \omega(A \times B \cap \{(t,z)\}^c),$$

$$\epsilon^+_{(t,z)}\omega(A \times B) := \epsilon^-_{(t,z)}\omega(A \times B) + \mathbf{1}_A(t)\mathbf{1}_B(z).$$

$$D_{(t,z)}F := F \circ \epsilon^+_{(t,z)} - F$$

**Theorem 6** *Let $F$ be bounded and $\mathcal{F}_1$-measurable. Then*

$$F = E(F) + \int_0^1 \int_{\mathbb{R}_0} [D_{(t,z)}F]^p \; [\mu - \pi](dt, dz),$$

*where $[D_{(\cdot,z)}F]^p$ is the predictable projection of $D_{(\cdot,z)}F$.*

# Generating information drifts

RECALL: $P_t(\cdot, A) = P(A) + \int_0^t \int_{\mathbb{R}_0} k_s(z, A)\ [\mu - \pi](ds, dz)$

**Theorem 7** *Let* $A \in \mathcal{F}$. *Then*

$$\begin{aligned} k_t(z, A) &= [D_{(t,z)}(P_{t+}(\omega, A))]^p \\ &= P_{t-}(\epsilon^{+}_{(t,z)}\omega, A) - P_{t-}(\omega, A) \end{aligned}$$

**Example:**

$$X_t = \int_0^t \int_{\mathbb{R}_0} \psi(s,z)[\mu - \pi](dr, dz)$$

$(\mathcal{F}_t^0)$ = filtration generated by $\mu$

$$\mathcal{G}_t^0 = \mathcal{F}_t^0 \vee \sigma(|X_1|) \qquad \text{(initial enlargement)}$$

Suppose $P(X_1 - X_t \in dx) \ll$ Lebesgue measure and

$$f(t,x) = \frac{P(X_1 - X_t \in dx)}{dx}$$

Then

$$P_t(\cdot, |X_1| \le c) = \int_0^c [f(t, y - X_t) + f(t, -y - X_t)] dy$$

and

$$P_{t+}(\epsilon^+_{(t,z)}\omega, |X_1| \le c)) = \int_0^c [f(t, y - X_t(\omega) - z) + f(t, -y - X_t(\omega) - z)] dy$$

Consequently,

$$\begin{aligned} k_t(z, |X_1| \le c) &= \int_0^c [f(t, y - X_{t-} - z) + f(t, -y - X_{t-} - z)] dy \\ &\quad -P_{t-}(\cdot, |X_1| \le c), \end{aligned}$$

$$\longrightarrow \delta_t(\omega, \omega') = \frac{\int_{\mathbb{R}_0} \psi(t,z) k_t(\omega, z, d\omega') \; d\nu(z)}{P_t(\omega, d\omega')}\Big|_{\sigma(|X_1|)}$$

**Lemma 2** *The information drift* $\eta_t$ *of* $X$ *relative to* $(\mathcal{G}_t)$ *is given by*

$$\int_{\mathbb{R}_0} \left[ \frac{f(t, |X_1| - X_{t-} - z) + f(t, -|X_1| - X_{t-} - z)}{f(t, |X_1| - X_{t-}) + f(t, -|X_1| - X_{t-})} - 1 \right] \psi(t,z)\nu(dz)$$

**Remark:**

a) If $\int_{\mathbb{R}_0} |\psi(t,z)| d\nu(z) < \infty \quad \Longrightarrow$ separate terms

b) This scheme works for many examples

# Conclusion

- enlargements of filtrations can be seen as random changes of measure
- variational calculus allows to derive explicit semimartingale decompositions with respect to enlarged filtrations
- on Wiener space: information drift $=$ logarithmic Malliavin trace of the conditional probability relative to the enlarging information
- on a Poisson space: information drift $=$ logarithmic Picard trace of the conditional probability relative to the enlarging information

Thanks for your attention!