Existence of the occupation density of perturbed gaussian process which has a covariance measure

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Outline

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- Main result: Extension for the criteria of existence of the occupation density.

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- Notion of covariance measure structure for square integrable stochastic processes (Kruk; Russo and Tudor 2006)
- Main result: Extension for the criteria of existence of the occupation density.
- Examples: Fractional Brownian motion and subractional Brownian motion.

Occupation density: Definition

- Let T a Borel subset of $[0,1]$ and f a measurable function $T \to \mathbb{R}$. The occupation density of f on T is the density with respect to Lebesgue measure of the occupation measure

$$A \longrightarrow \int_T 1_A(f(s))\,ds, \quad A \in \mathcal{B}(\mathbb{R}).$$

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- On a probability space $(\Omega, \mathcal{F}, P)$, we say that a continuous stochastic process $(X_t, 0 \le t \le 1)$ has an occupation density on $(T, C) \in \mathcal{B}([0,1]) \otimes \mathcal{F}$ if for almost all $w \in C$, $X_.(w)$ has an occupation density on T.

Criteria of occupation densities (Geman and Horowitz)

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- Theorem (1): (Geman and Howoritz)

  Consider a probability space $(\Omega, \mathcal{F}, P)$. Let T a Borel subset of $[0,1]$, $A \in \mathcal{F}$, $(X_t, t \in [0,1])$ a continuous stochastic process. Then the process X possesses a square integrable occupation density on (A, T) if and only if

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_T \int_T P(A \cap \{|X_t - X_s| < \varepsilon) dsdt < \infty \qquad (1)$$

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  - Let F be a bounded r.v. belongs to $\mathbb{D}^{1,1}$ and $h \in \mathcal{H}$. Then

$$E(F\delta(h)) = E\langle DF, h\rangle$$

.

Criteria of occupation densities (Imkeller and Nualart)

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- Proposition
  Let $U \in D^{2,2}$, $h \in \mathcal{H}$ *and* $F \in D^{1,1}$. *Suppose that*

$$\langle DU, h\rangle_{\mathcal{H}} > 0 \quad a.s \text{ on } \{F \neq 0\}$$

and let $f \in C_b^{\infty}(\mathbb{R})$. *Then*

$$|E(f'(U)F)| \leq \|f\|_{\infty} E\left(\frac{|\delta(h)F| + |\langle DF, h\rangle_{\mathcal{H}}|}{\langle DU, h\rangle_{\mathcal{H}}} + \frac{F\langle D^2U, h\otimes h\rangle_{\mathcal{H}\otimes\mathcal{H}}}{\langle DU, h\rangle_{\mathcal{H}}^2}\right).$$

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- Theorem (2): (Imkeller and Nualart)

  Let $(X_t, t \in [0,1])$ *be a stochastic continuous process such that* $X_t \in \mathbb{D}^{2,2}$ *for any* $t \in [0,1]$. *Let* $F \in \mathbb{D}^{1,1}$ *be a bounded random variable. Assume that there exist a constant* $\delta > 0$, *a subinterval* $T \subset [0,1]$ *and a bounded and measurable function* $\beta : T \longrightarrow \mathbb{R}$, *such that*

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  - *a)* For every $0 \leq s \leq t \leq 1$

$$\left\langle D(X_t - X_s), \beta 1_{(s,t]} \right\rangle_{\mathcal{H}} \geq \delta(t-s) \text{ on } \{F \neq 0\}.$$

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  - *b)* $\int_T \int_T \frac{E\left|\int_s^t \beta_r D_r F dr\right|}{|t-s|} dsdt < \infty$

Criteria of occupation densities (Imkeller and Nualart)

- c) $\int_T \int_T \frac{E\left|F \int_s^t \int_s^t \beta_u \beta_r D_u D_r (X_t - X_s) dr du\right|}{(t-s)^2} dsdt < \infty.$

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- They proved the existence of an occupation density in the following cases:
  - The first nonadapted process considered is the form

$$X_t = W_t + \int_0^t u_s ds, \quad 0 \leq t \leq 1,$$

where u belong to the space $L^{2,2} = D^{2,2}(\mathcal{H})$.

Criteria of occupation densities (Imkeller and Nualart)

- The second process studied is the form

$$X_t = \int_0^t u_s dW_s, \quad 0 \le t \le 1,$$

With some smoothness and integrability hypotheses on the process u.

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- Proposition

For the case of fBm with $H < \frac{1}{2}$, we assume only that $u \in \mathbb{L}^{2,2}$ and we prove that the process defined by

$$X_t = B_t^H + \int_0^t u_s ds$$

possesses a square integrable occupation density.

Existence of the occupation density for fBm.

Proof. By using the theorem 2, it suffices to check that

$$\langle D^W(B_t^H - B_s^H), 1_{(s,t]}\rangle \geq C(t-s) \tag{2}$$

where C is a positif constant.
By integral representation of B^H we have

$$B_t^H = \int_0^t k(t,r)dW_r$$

where $k(t,r) = C_H(t-r)^{H-1/2} + r^{H-1/2}F(\frac{t}{s})$, with

$$F(z) = C_H(\frac{1}{2} - H)\int_0^{z-1} \theta^{H-\frac{3}{2}}(1-(\theta+1)^{H-\frac{1}{2}})d\theta$$

Existence of the occupation density for fBm.

- This implies that

$$\begin{aligned}
\langle D^W(B_t^H - B_s^H), 1_{(s,t]}\rangle &= \int_s^t k(t,r)dr \\
&\geq C_H \int_s^t (t-r)^{H-1/2} dr \\
&= \frac{C_H}{H+\frac{1}{2}}(t-s)^{H+1/2} \\
&\geq \frac{C_H}{H+\frac{1}{2}}(t-s).
\end{aligned}$$

Thus (2) is satisfied.

Occupation density for gaussian process which has a covariance measure.

- Let $(Y_t, t \in [0,1])$ be a continuous zero-mean square integrable stochastic process with covariance

$$R(s,t) = E(Y_t Y_s)$$

The covariance function R defines a $\mathbb{R}$-valued finitely additive function (noted μ) on the algebra of finite disjoint rectangles included in $[0,1]^2$ in this way:

$$\mu((a,b] \times (c,d]) = R(b,d) + R(a,c) - R(a,d) - R(b,c)$$

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Definition

We say that the process Y has a covariance measure if the measure μ extends to the Borel σ-field $\mathcal{B}([0,1]^2)$ to a signed σ-finite measure.

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Theorem

Let $(X_t, t \in [0,1])$ be a stochastic continuous process, F a random variable in $D^{1,2}$ and σ a positive constant, such that

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Theorem

Let $(X_t, t \in [0,1])$ be a stochastic continuous process, F a random variable in $D^{1,2}$ and σ a positive constant, such that

- *A)* For every $0 \le s \le t \le 1$

$$\left\langle D(X_t - X_s), 1_{(s,t]} \right\rangle_{\mathcal{H}} \ge \sigma E(Y_t - Y_s)^2 \text{ on } \{F \neq 0\}$$

Occupation density for gaussian process which has a covariance measure.

- *B*) $\int_0^1 \int_0^1 \frac{E\left|\left\langle DF, 1_{(s,t]}\right\rangle_{\mathcal{H}}\right|}{E(Y_t - Y_s)^2} dsdt < \infty$

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- $C)$ $\int_0^1 \int_0^1 \frac{E\left|F\left\langle DD(X_t - X_s), 1_{(s,t]}^{\otimes 2}\right\rangle_{\mathcal{H}\otimes\mathcal{H}}\right|}{(E(Y_t - Y_s)^2)^2} dsdt < \infty.$

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Remark

If Y a continuous gaussian process satisfied the condition given by Geman (1976):

$$\int_0^1 \int_0^1 \frac{1}{(E(Y_t - Y_s)^2)^{1/2}} < \infty.$$

Then Y possesses a square integrable occupation density.

Occupation density for gaussian process which has a covariance measure.

- Recall that the bifBm $B^{H,K}$ is a centered Gaussian process, starting from zero, with covariance function

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right)$$

where the parameters H, K are such that $H \in (0,1)$ and $K \in (0,1]$.

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where the parameters H, K are such that $H \in (0,1)$ and $K \in (0,1]$.
- In the case $K = 1$ we retrieve the fractional Brownian motion while the case $K = 1$ and $H = \frac{1}{2}$ corresponds to the standard Brownian motion.

Occupation density for gaussian process which has a covariance measure.

- The process $B^{H,K}$ is HK-selfsimilar but it has no stationary increments. It has Hölder continuous paths of order $\delta < HK$ and its paths are not differentiable.

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- The process $B^{H,K}$ is HK-selfsimilar but it has no stationary increments. It has Hölder continuous paths of order $\delta < HK$ and its paths are not differentiable.
- An interesting property of it is the fact that its quadratic variation in the case $2HK = 1$ is similar to that of the standard Brownian motion, i.e. $[B^{H,K}]_t = cst. \times t$ and therefore especially this case $(2HK = 1)$ is very interesting from the stochastic calculus point of view.

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- Proposition

  Assume that $2HK = 1$*. If* $\{u_t, t \in [0,1]\}$ *a process such that,* $u_t \in D^{2,2}$ *for any* $t \in [0,1]$ *and satisfied*

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  1)

$$E\left(\int_0^1 \int_0^1 \left(\int_0^1 |D_\alpha u_r|^2 dr\right) \mu(d\alpha, d\beta)\right) < \infty$$

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1)

$$E\left(\int_0^1\int_0^1\left(\int_0^1 |D_\alpha u_r|^2 dr\right)\mu(d\alpha, d\beta)\right) < \infty$$

2) and

$$E\left(\int_{[0,1]^4}\left(\int_0^1 |D_\theta D_\alpha u_r|^2 dr\right)\mu(d\alpha, d\beta)\mu(d\theta, \gamma)\right) < \infty$$

where
$\mu(d\alpha, d\beta) = \frac{2H(K-1)}{2^K}(\beta^{2H} + \alpha^{2H})^{K-2}(\alpha\beta)^{2H-1} d\alpha d\beta.$

Occupation density for gaussian process which has a covariance measure.

- Then the perturbed bifractional Brownian motion

$$B_t^{H,K} + \int_0^1 u_s ds$$

possesses a square integrable occupation density.

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- Let ρ be a constant in $(1/2, 1)$, then there exist a real δ in $(1/2, 1)$ such that

$$\frac{1+\delta}{2(2-\delta)} < \rho < \frac{1}{2-\delta} \tag{3}$$

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- We consider a continuous gaussian process $(Y_t, t \in [0,1])$ which has a positive finite covariance measure μ on $\mathcal{B}([0,1]^2)$ satisfied the following condition:

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- We consider a continuous gaussian process $(Y_t, t \in [0,1])$ which has a positive finite covariance measure μ on $\mathcal{B}([0,1]^2)$ satisfied the following condition:

- μ is continuous absolutely with respect to the Lebesgue measure on $\mathbb{R}^2$.

$$d\mu(d\theta, d\gamma) = g(\theta, \gamma) d\theta d\gamma.$$

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- Theorem (3)

  Let $\{u_t, t \in [0,1]\}$ *a process satisfied,* $u_t \in D^{2,2}$ *for any* $t \in [0,1]$*. Moreover we suppose that*

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$$E(Y_t - Y_s)^2 \geq (t-s)^{2\rho}$$

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2)

$$E\left(\int_0^1 \int_0^1 \left(\int_0^1 |D_\alpha u_r|^{\frac{1}{1-(2-\delta)\rho}} dr\right)^\lambda \mu(d\alpha, d\beta)\right)^2 < \infty$$

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  $$E\left(\int_0^1 \int_0^1 (\int_0^1 |D_\theta D_\alpha u_r|^{\frac{1}{1-(2-\delta)\rho}} dr) d\alpha d\theta\right) < \infty.$$

  where $\lambda = \frac{1-(2-\delta)\rho}{1-\delta}$.

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Then, the stochastic process X defined by

$$X_t = Y_t + \int_0^t u_s ds$$

possesses a square integrable occupation density.

Fractional Brownian motion

- ***Y* is a fractional Brownian motion B^H with $H > \frac{1}{2}$:** If $H > 1/2$, We recall that its covariance equals, for every $s;\ t \in [0;1]$

$$R(s;t) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$$

In this case: $\frac{\partial^2 R}{\partial s \partial t}(s,t) = 2H(2H-1)|t-s|^{2H-2}$ in the sense of distributions. Since R vanishes on the axes, we have

$$R(s;t) = \int_0^t \int_0^s \frac{\partial^2 R}{\partial s \partial t}(s,t) dsdt = \mu((0,s] \times (0,t])$$

Fractional Brownian motion

- We conclude that B^H has a positive covariance measure of this form

$$\mu(ds, dt) = 2H(2H-1)|t-s|^{2H-2}dsdt$$

and since $E(B_t^H - B_s^H)^2 = |t-s|^{2H}$, then under same hypothesis in theorem (3) on the process u we obtain that

$$B_t^H + \int_0^t u_s ds$$

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- SubfBm(Bojdecki, Gorostiza and Talarczyk: (2004)) A centered Gaussian process $(Z_t^H, t \geq 0)$ is called subfBm of Hurst parameter $H \in (0,1)$, if

$$\begin{aligned} C_h(t,s) &= E(Z_t^H Z_s^H) \\ &= |t|^{2H} + |s|^{2H} - \frac{1}{2}\left(|t+s|^{2H} + |t-s|^{2H}\right) \end{aligned}$$

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- Z^H is not a Markov process if $2H \neq 1$.
- Z^H is not a semimartingale if $2H \neq 1$

Sub-fractional Brownian motion

- If $2H > 1$, for every $s \leq t$

$$(2 - 2^{2H-1})(t - s)^{2H} \leq E\left(Z_t^H - Z_s^H\right)^2 \leq (t - s)^{2H}$$

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- The covariance of subfBm ca be written as

$$\begin{aligned} C_H(t,s) &:= \mu((0,s] \times (0,t]) \\ &= \alpha_H \int_0^t \int_0^s (|r-u|^{2H-2} - |r+u|^{2H-2}) drdu \end{aligned}$$

where $\alpha_H = \frac{2H(2H-1)}{2}$.
Hence Z has a positive covariance measure satisfied the same conditions as in theorem (3).

Sub-fractional Brownian motion

- Therefore under the same conditions in theorem (3) on the process u, we see that

$$Z_t^H + \int_0^t u_s ds$$

possesses a square integrable occupation density.