Optimal stochastic control and BSDEs in infinite dimensions

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Plan

Stochastic integrals and evolution equations in Hilbert spaces.

Forward-backward stochastic differential systems.

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Kolmogorov partial differential equations (PDEs).

Forward-backward stochastic equations and nonlinear PDEs.

Wiener process and stochastic integrals in Hilbert spaces

A cylindrical Wiener process in a Hilbert space K is a family of random variables

$$W = \{W_t^k,\ t \geq 0,\ k \in K\}$$

such that

- for each $k \in K$, $\{W_t^k,\ t \geq 0\}$ is a real centered Wiener process;
- for each $t \geq 0$, the mapping $k \to W_t^k$ is linear and satisfies

$$\mathbb{E}\,[W_t^k\,W_s^h] = (t \wedge s)\,\langle k, h\rangle.$$

It can be defined by the formula

$$W_t^k = \sum_{n=0}^{\infty} \beta_t^n \langle k, e_n\rangle,$$

where (e_n) is a basis of K and (β^n) an independent sequence of real standard Wiener processes. *Formally:*

$$W_t = \sum_{n=0}^{\infty} \beta_t^n e_n.$$

Stochastic integrals with values in another Hilbert space H can be defined:

$$I_t = \int_0^t \Phi_s \, dW_s, \qquad t \geq 0, \tag{1}$$

where $\{\Phi_t, \, t \geq 0\}$ is a process with values in (a subspace of) $L(K, H)$.

For "simple" $\{\Phi_t, \, t \geq 0\}$ we have the Ito isometry

$$\mathbb{E}\,|I_t|^2 = \mathbb{E} \int_0^t |\Phi_s|^2_{L_2(K,H)} ds.$$

[$\Phi \in L_2(K, H)$, the Hilbert-Schmidt class, if $|\Phi|^2_{L_2(K,H)} = \sum_{n=0}^{\infty} |\Phi e_n|^2_H < \infty$.]

If $\{\Phi_t, \, t \geq 0\}$ is a progressive process in $L_2(K, H)$ satisfying

$$\int_0^t |\Phi_s|^2_{L_2(K,H)} ds < \infty, \qquad \mathbb{P} - a.s., \; t \geq 0,$$

then $\{I_t, \, t \geq 0\}$ defined in (1) is a local martingale in H.

Remark. Suppose $K = L^2(0, 1)$. For $t \geq 0$ and $x \in [0, 1]$ one can define random variables

$$\mathcal{W}([0, t] \times [0, x]) = W_t^k, \quad \text{where} \quad k = 1_{[0,x]}.$$

The space-time white noise is the (distributional) derivative

$$\dot{\mathcal{W}}(t, x) = \frac{\partial^2}{\partial x \partial t} \mathcal{W}([0, t] \times [0, x]).$$

Stochastic evolution equations in Hilbert spaces

$$\begin{cases} dX_t = AX_t\,dt + F(t,X_t)\,dt + G(t,X_t)\,dW_t, & t \in [0,T], \\ X_0 = x \in H, \end{cases}$$

where:

• W is a cylindrical Wiener process in a Hilbert space K;

• the unknown process $X = \{X_t,\ 0 \leq t \leq T\}$ takes values in another Hilbert space H;

• A is a linear operator in H;

• F takes values in H and G takes values in $L(K,H)$.

Motivation: stochastic PDEs

Stochastic heat equation on the interval $\xi \in [0,1]$:

$$\begin{cases} \dfrac{\partial}{\partial t} y(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} y(t,\xi) + f(y(t,\xi)) + g(y(t,\xi))\,\dot{\mathcal{W}}(t,\xi), \\ y(t,0) = y(t,1) = 0, \\ y(0,\xi) = x_0(\xi), \end{cases} \tag{2}$$

where $t \in [0,T]$ and $\dot{\mathcal{W}}$ is space-time white noise in $L^2(0,1)$.

Suppose $f, g \in C^1(\mathbb{R})$ with bounded derivatives, g bounded.

Reformulation as a controlled evolution equation: we define

$$H = K = L^2(0,1), \qquad X_t = y(t,\cdot).$$

We set, for $x(\cdot) \in H = L^2(0,1)$,

$$F(x)(\cdot) = f(x(\cdot)), \quad G(x)(\cdot) = g(x(\cdot)), \quad Ax = \frac{\partial^2 x}{\partial \xi^2},$$

with domain $\mathrm{dom}(A) = H^2(0,1) \cap H^1_0(0,1)$. Then (2) becomes

$$dX_t = AX_t\,dt + F(X_t)\,dt + G(X_t)\,dW_t,$$

with initial condition $X_0 = x_0$, where W is a cylindrical Wiener process in K.

Assume the following *general assumptions*:

- A generates a strongly continuous semigroup $\{e^{tA},\, t \geq 0\}$ in H.

- $F : [0,T] \times H \to H$ satisfies, for $t \in [0,T]$, $x, y \in H$,

$$|F(t,x)| \leq C(1 + |x|), \;\; |F(t,x) - F(t,y)| \leq C|x - y|,$$

- $G : [0,T] \times H \to L(K,H)$ is bounded and for $s > 0$, $t \in [0,T]$, $x, y \in H$,

$$\begin{aligned} &|e^{sA}G(t,x)|_{L_2(K,H)} \leq C\, s^{-\gamma}(1 + |x|), \\ &|e^{sA}G(t,x) - e^{sA}G(t,y)|_{L_2(K,H)} \leq C\, s^{-\gamma}|x - y|, \end{aligned}$$

  for some $\gamma \in [0, 1/2)$.

For every $t \in [0,T]$ and $x \in H$ there exists a unique solution of the equation

$$\begin{cases} dX_s = AX_s\, ds + F(s, X_s)\, ds + G(s, X_s)\, dW_s, & s \in [t,T] \subset [0,T], \\ X_t = x \in H, \end{cases}$$

i.e. an adapted continuous process satisfying

$$X_s = e^{(s-t)A}x + \int_t^s e^{(r-t)A}F(r, X_r)\, dr + \int_t^s e^{(r-t)A}G(r, X_r)\, dW_r, \quad s \in [t,T].$$

The solution is a Markov process in H denoted $\{X_s^{t,x},\, 0 \leq t \leq s \leq T,\, x \in H\}$.

The backward equation

With X as before we also consider the backward differential equation for the unknown process $\{(Y_s, Z_s),\ s \in [t, T]\}$:

$$\begin{cases} dY_s = \psi(X_s^{t,x}, Y_s, Z_s)\, ds + Z_s\, dW_s, & s \in [t, T], \\ Y_T = \phi(X_T^{t,x}). \end{cases}$$

Y is real and Z takes values in K^*, ψ and ϕ are functions such that

- $\phi : H \to \mathbb{R}$ is Lipschitz and $\psi : H \times \mathbb{R} \times K^* \to \mathbb{R}$ satisfies

$$|\psi(x, y_1, z_1) - \psi(x, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|),$$
$$|\psi(x_1, y, z) - \psi(x_2, y, z)| \leq C|x_2 - x_1|(1 + |z|)(1 + |x_1| + |x_2| + |y|)^m.$$

Pardoux and Peng 90 proved that there exists a unique $(\mathcal{F}_s)$-adapted solution satisfying

$$\mathbb{E} \sup_{s \in [t,T]} |Y_s|^2 < \infty, \qquad \mathbb{E} \int_t^T |Z_s|^2 ds < \infty.$$

We denote $Y_s = Y_s^{t,x}$, $Z_s = Z_s^{t,x}$.

$Y_t^{t,x}$ is deterministic. We set

$$v(t, x) = Y_t^{t,x}, \qquad t \in [0, T], x \in H.$$

Then, for some Borel function $\zeta(t, x)$, for $0 \leq t \leq s \leq T$, $x \in H$,

$$Y_s^{t,x} = v(s, X_s^{t,x}), \qquad Z_s^{t,x} = \zeta(s, X_s^{t,x}).$$

Regularity of v and identification of Z

Given $(t,x) \in [0,T] \times H$ we set $v(t,x) = Y_t^{t,x}$, where

$$\begin{cases} dX_s^{t,x} = AX_s^{t,x}\, ds + F(s, X_s^{t,x})\, ds + G(s, X_s^{t,x})\, dW_s, \quad s \in [t,T] \subset [0,T], \\ X_t^{t,x} = x, \\ dY_s^{t,x} = \psi(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\, ds + Z_s^{t,x}\, dW_s, \\ Y_T^{t,x} = \phi(X_T^{t,x}). \end{cases}$$

Theorem *If $F(t,\cdot), G(t,\cdot), \phi, \psi \in C^1$ then $v(t,\cdot) \in C^1$ with polynomial growth and*

$$Z_s^{t,x} = \nabla v(s, X_s^{t,x})\, G(s, X_s^{t,x}), \qquad 0 \le t \le s \le T,\, x \in H.$$

For $H = \mathbb{R}^n$ see Pardoux-Peng LNCIS 92. For the general case F.-Tessitore AOP 02.

Idea of the proof. One starts from

$$Y_s^{t,x} = v(s, X_s^{t,x}),$$

and takes the Malliavin derivative:

$$D_\sigma Y_s^{t,x} = \nabla v(s, X_s^{t,x})\, D_\sigma X_s^{t,x}, \qquad 0 \le t \le \sigma \le s \le T.$$

Taking Malliavin derivatives in the forward-backward system one finds

$$D_\sigma Y_s^{t,x} \to Z_s^{t,x}, \qquad D_\sigma X_s^{t,x} \to G(s, X_s^{t,x}),$$

as $\sigma \uparrow s$.

Optimal stochastic control problems

$$\begin{cases} dX_s = AX_s\,ds + F(s,X_s)\,ds + G(s,X_s)\,dW_s, & s\in[t,T]\subset[0,T],\\ X_t = x\in H, \end{cases}$$

A control process is an adapted process u with values in U. To each control we associate a cost as follows. The process

$$W_s^u = W_s - \int_t^s R(X_r,u_r)\,dr, \qquad R: H\times U\to K \text{ bounded}$$

is Wiener under $\mathbb{P}^u = \rho\,\mathbb{P}$,

$$\rho = \exp\left(\int_0^T R(X_s,u_s)^*\,dW_s - \frac{1}{2}\int_0^T |R(X_s,u_s)|^2\,ds\right).$$

The cost is

$$J(t,x,u(\cdot)) = \mathbb{E}^u \int_t^T L(X_s,u_s)\,ds + \mathbb{E}^u\,\phi(X_T),$$

with L,ϕ real functions. X is the (unique in law) solution of

$$\begin{cases} dX_s = AX_s\,ds + F(s,X_s)\,ds + G(s,X_s)R(X_s,u_s)\,ds + G(s,X_s)\,dW_s^u, & s\in[t,T],\\ X_t = x. \end{cases}$$

Examples: controlled stochastic PDEs

Example 1. Controlled stochastic heat equation driven by white noise:

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(t,\xi) + f(y(t,\xi)) + g(y(t,\xi))\,[u(t,\xi) + \dot{\mathcal{W}}(t,\xi)], \\ y(t,0) = y(t,1) = 0, \\ y(0,\xi) = x_0(\xi), \end{cases}$$

Here $(t,\xi) \in [0,T] \times [0,1]$ and u is a real random field.

Example 2. Heat equation with boundary noise and boundary control:

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(t,\xi) + f(y(t,\xi)), \\ \dfrac{\partial y}{\partial \xi}(t,0) = u_t^1 + \dfrac{\partial W_t^1}{\partial t}, \\ \dfrac{\partial y}{\partial \xi}(t,1) = u_t^2 + \dfrac{\partial W_t^2}{\partial t}, \\ y(0,\xi) = x_0(\xi), \end{cases}$$

Here (W^1, W^2) is Wiener in $\mathbb{R}^2$ and $u = (u^1, u^2)$.

See Debussche-F.-Tessitore 07.

BSDE approach to control problems

Control problem starting at (t,x): minimize

$$J(t,x,u(\cdot)) = \mathbb{E}^u \int_t^T L(X_s,u_s)\,ds + \mathbb{E}^u\,\phi(X_T),$$

where

$$\begin{cases} dX_s = AX_s\,ds + F(s,X_s)\,ds + G(s,X_s)R(X_s,u_s)\,ds + G(s,X_s)\,dW_s^u, \quad s\in[t,T], \\ X_t = x \in H. \end{cases}$$

Value function: $v(t,x) = \inf_{u(\cdot)} J(t,x,u(\cdot))$.

Solve the forward-backward system

$$\begin{cases} dX_s = AX_s\,ds + F(s,X_s)\,ds + G(s,X_s)\,dW_s, \quad s\in[t,T]\subset[0,T], \\ X_t = x, \\ dY_s = \psi(X_s,Z_s)\,ds + Z_s\,dW_s, \\ Y_T = \phi(X_T). \end{cases}$$

where

$$\psi(x,z) = -\inf_{u\in U}[L(x,u) + z\,R(x,u)].$$

We assume the infimum is achieved at some measurable

$$u = \gamma(x,z).$$

From the backward equation

$$Y_t = \phi(X_T^{t,x}) - \int_t^T [Z_s R(X_s, u_s) + \psi(X_s, Z_s)]\, ds - \int_t^T Z_s\, dW_s^u$$

adding and subtracting $\int_t^T L(X_s, u_s)\, ds$ and taking $\mathbb{E}^u$:

$$Y_t = J(t, x, u(\cdot)) - \int_t^T [L(X_s, u_s) + Z_s R(X_s, u_s) + \psi(X_s, Z_s)]\, ds.$$

Since [...] ≥ 0 it follows that $Y_t \leq J(t, x, u(\cdot))$ and we have equality provided

$$u_s = \gamma(X_s, Z_s).$$

So this control is optimal, the value function is

$$v(t, x) = Y_t = Y_t^{t,x},$$

and the optimal control is given by a feedback law

$$u_s = \gamma(X_s, \zeta(s, X_s)) =: \underline{u}(s, X_s).$$

If v is regular, then then feedback law is related to the gradient of the value function:

$$u_s = \gamma(X_s, \nabla v(s, X_s)\, G(s, X_s)).$$

Problem: characterize $v(t, x) = Y_t^{t,x}$ (as the solution of a PDE).

Extensions and complements

1. Given X, consider again

$$\begin{cases} dY_s = \psi(X_s^{t,x}, Y_s, Z_s)\, ds + Z_s\, dW_s, & s \in [t,T], \\ Y_T = \phi(X_T^{t,x}). \end{cases}$$

More general conditions have been considered on ψ:
• $\psi(x,y,z)$ Lipschitz in z and dissipative in y: Briand-Confortola AMO 07.
• $\psi(x,y,z)$ quadratic in z: Briand-Confortola SPA 07.

2. General equation

$$\begin{cases} dY_t = -B\, Y_t\, dt + f(\omega, t, Y_s, Z_s)\, dt + Z_t\, dW_t, & t \in [0,T], \\ Y_T = \xi(\omega), \end{cases}$$

with Y taking values in a Hilbert space V (and Z in $L_2(K,V)$).

• Solvability:
Hu-Peng SAA 91 (+ stochastic maximum principle), Pardoux-Rascanu Stochastics 99, Confortola SPA 06.
• Applications to stochastic games (with an infinite number of players):
F.-Hu AMO 07.

3. "Fully coupled" FBDEs: Ma-Yong (book) 99, Guatteri JSAA 07.

4. Equations on $[0,\infty)$ etc.

Kolmogorov equations on a Hilbert space

Consider again the Markov process $\{X_s^{t,x},\ 0 \le t \le s \le T,\ x \in H\}$ in H defined by the equation

$$\begin{cases} dX_s = AX_s\,ds + F(s,X_s)\,ds + G(s,X_s)\,dW_s, & s \in [t,T] \subset [0,T], \\ X_t = x \in H. \end{cases}$$

Define its transition semigroup $\{P_{t,s} : 0 \le t \le s \le T\}$ and its (formal) generator $\{\mathcal{L}_t : 0 \le t \le T\}$, acting on for bounded $\phi : H \to \mathbb{R}$:

$$P_{t,s}[\phi](x) = \mathbb{E}\,\phi(X_s^{t,x}), \qquad x \in H,$$

$$\mathcal{L}_t[\phi](x) \;=\; \frac{1}{2}\text{Trace}\left(G(t,x)G(t,x)^*\,\nabla^2\phi(x)\right) + \langle Ax + F(t,x), \nabla\phi(x)\rangle.$$

Then the function $\quad u(t,x) = P_{t,T}[\phi](x),$
is a candidate solution of the Kolmogorov equation

$$\frac{\partial u(t,x)}{\partial t} + \mathcal{L}_t[u(t,\cdot)](x) = 0, \qquad u(T,x) = \phi(x), \qquad t \in [0,T],\ x \in H.$$

More generally, the solution of the linear nonhomogeneous equation

$$\frac{\partial u(t,x)}{\partial t} + \mathcal{L}_t[u(t,\cdot)](x) = f(t,x), \qquad u(T,x) = \phi(x), \qquad t \in [0,T],\ x \in H.$$

has the candidate solution given by the variation of constants formula:

$$u(t,x) = P_{t,T}[\phi](x) - \int_t^T P_{t,s}\left[f(s,\cdot)\right](x)\,ds.$$

BSDEs and parabolic equations on a Hilbert space

Given (t,x) and real functions ϕ,ψ, solve the forward-backward system

$$\begin{cases} dX_s^{t,x} = AX_s^{t,x}\,ds + F(s,X_s^{t,x})\,ds + G(s,X_s^{t,x})\,dW_s, \qquad s\in[t,T]\subset[0,T], \\ X_t^{t,x} = x, \\ dY_s^{t,x} = \psi(X_s^{t,x},Y_s^{t,x},Z_s^{t,x})\,ds + Z_s^{t,x}\,dW_s, \\ Y_T^{t,x} = \phi(X_T^{t,x}), \end{cases}$$

and set $v(t,x) = Y_t^{t,x}$. Then v is a candidate solution of the semilinear parabolic PDE:

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} + \mathcal{L}_t[v(t,\cdot)](x) = \psi(x, v(t,x), \nabla v(t,x) G(t,x)), \\ \\ v(T,x) = \phi(x), \qquad t\in[0,T],\ x\in H. \end{cases}$$

This was first proved in Peng 92, Pardoux-Peng LNCIS 92 if $H=\mathbb{R}^n$.

Parabolic equations on a Hilbert space. The notion of mild solution

$$\frac{\partial v(t,x)}{\partial t} + \mathcal{L}_t[v(t,\cdot)](x) = \psi(x, v(t,x), \nabla v(t,x)G(t,x)), \qquad v(T,x) = \phi(x).$$

• Classical solutions (at least $u \in C^{1,2}((0,T) \times H)$):
Barbu-Da Prato (book, 1983), Da Prato-Zabczyk (3 books).

• Viscosity solutions (only $u \in C((0,T) \times H)$):
P.L. Lions (1988, 1989a, 1989b), Crandall-Kocan-Święch (1993/94), Święch (1994), Kocan-Święch (1995), Gozzi-Rouy-Święch (2000), Gozzi-Święch (2000).

• The notion of mild solution. In analogy with the variation of constants formula, we call v a mild solution if it satisfies, for $t \in [0,T]$, $x \in H$,

$$v(t,x) = P_{t,T}[\phi](x) - \int_t^T P_{t,s}\left[\psi(\cdot, v(s,\cdot), \nabla v(s,\cdot)G(s,\cdot))\right](x)\, ds.$$

Here we require $v(t,\cdot) \in C^1$ with polynomial growth.

Analytic approach to mild solutions: Da Prato-Cannarsa (1991, 1992), Gozzi (1995, 1996), Cerrai (LNM 1762), Masiero (2005).

Theorem (F.-Tessitore AOP 02). *Suppose that* $F(t,\cdot), G(t,\cdot), \phi, \psi \in C^1$ *and consider the FBSDE*

$$\begin{cases} dX_s^{t,x} = AX_s^{t,x}\, ds + F(s, X_s^{t,x})\, ds + G(s, X_s^{t,x})\, dW_s, \quad s \in [t,T] \subset [0,T], \\ X_t^{t,x} = x, \\ dY_s^{t,x} = \psi(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\, ds + Z_s^{t,x}\, dW_s, \\ Y_T^{t,x} = \phi(X_T^{t,x}), \end{cases}$$

Setting $v(t,x) = Y_t^{t,x}$, *then* $v(t,\cdot) \in C^1$ *with polynomial growth and* v *is the unique mild solution of the PDE:*

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} + \mathcal{L}_t[v(t,\cdot)](x) = \psi(x, v(t,x), \nabla v(t,x) G(t,x)), \\ \\ v(T,x) = \phi(x), \qquad t \in [0,T],\ x \in H. \end{cases} \tag{3}$$

In particular, in control problems this characterizes the value function as the unique solution to the Hamilton-Jacobi-Bellman equation (3).

Under the ellipticity condition $|G(t,x)^{-1}| \leq C$ less regularity is required on ϕ and ψ: see F.-Tessitore Stochastics 04.

Sketch of the proof. Existence. Mild solution:

$$v(t,x) = P_{t,T}[\phi](x) - \int_t^T P_{t,s}\left[\psi(\cdot, v(s,\cdot), \nabla v(s,\cdot)G(s,\cdot))\right](x)\, ds.$$

Fix t, x and denote $X_s = X_s^{t,x}$, $X_s = X_s^{t,x}$, $X_s = X_s^{t,x}$, $s \in [t,T]$.
By a previous result, $v(s,\cdot) \in C^1$ and

$$Y_s = v(s, X_s), \qquad Z_s = \nabla v(s, X_s)\, G(s, X_s).$$

Then

$$\begin{aligned} &P_{t,s}[\psi(\cdot, v(s,\cdot), \nabla v(s,\cdot)G(s,\cdot))](x) \\ &\quad = \mathbb{E}\,\psi\big(X_s, v(s,X_s), \nabla v(s,X_s)G(s,X_s)\big) \\ &\quad = \mathbb{E}\,\psi\big(X_s, Y_s, Z_s\big), \end{aligned}$$

and $P_{t,T}[\phi](x) = \mathbb{E}\,\phi(X_T)$.
Next we recall the backward equation

$$Y_t + \int_t^T Z_s dW_s = \phi(X_T) - \int_t^T \psi(X_s, Y_s, Z_s) ds,$$

and we take expectation:

$$\begin{aligned} v(t,x) = Y_t &= \mathbb{E}\,\phi(X_T) - \int_t^T \mathbb{E}\,\psi(X_s, Y_s, Z_s) ds \\ &= P_{t,T}[\phi](x) - \int_t^T P_{t,s}[\psi(\cdot, v(s,\cdot), \nabla v(s,\cdot)G(s,\cdot))](x)\, ds. \end{aligned}$$