Approximate controllability for linear SDEs

Dan Goreac
Université de Bretagne Occidentale, Brest, France

La Londe, 10-14 september, 2007

$$dy(t) = (Ay(t) + Bu(t))\,dt + (Cy(t) + Du(t))\,dW(t),$$

Outline

- Approximate controllability in finite dimensions
- Approximate controllability in infinite dimensions

Introduction

- $(\Omega, \mathcal{F}, P)$ a complete probability space

Introduction

- $(\Omega, \mathcal{F}, P)$ a complete probability space
- $(W(t), t \geq 0)$ 1-dimensional Brownian motion on this space

Introduction

- $(\Omega, \mathcal{F}, P)$ a complete probability space
- $(W(t), t \geq 0)$ 1-dimensional Brownian motion on this space
- $(\mathcal{F}_t)_{t\geq 0}$ the completed natural filtration generated by W

Introduction

- $(\Omega, \mathcal{F}, P)$ a complete probability space
- $(W(t), t \geq 0)$ 1-dimensional Brownian motion on this space
- $(\mathcal{F}_t)_{t\geq 0}$ the completed natural filtration generated by W
- $\mathcal{A} = \mathcal{A}(\Omega, \mathcal{F}, P; W)$ the set of all $(\mathcal{F}_t)$-progressively measurable processes $v(\cdot)$ taking their values in $\mathbb{R}^d$ such that $E\left[\int_0^T |v(s)|^2 ds\right] < \infty$ for all $T > 0$.

- $$\begin{aligned} dy(t) &= (Ay(t) + Bu(t))\,dt + (Cy(t) + Du(t))\,dW(t), \\ 0 &\le t \le T, \end{aligned} \tag{1}$$

  $u(\cdot) \in \mathcal{A}$, where $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $B, D \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$.

- $$\begin{aligned} dy(t) &= (Ay(t) + Bu(t))\,dt + (Cy(t) + Du(t))\,dW(t), \\ 0 &\le t \le T, \end{aligned} \tag{1}$$

  $u(\cdot) \in \mathcal{A}$, where $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $B, D \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$.
- The initial condition is

$$y(0) = x \in \mathbb{R}^n. \tag{2}$$

- $$\begin{aligned} dy(t) &= (Ay(t) + Bu(t))\, dt + (Cy(t) + Du(t))\, dW(t), \\ 0 &\le t \le T, \end{aligned} \tag{1}$$

  $u(\cdot) \in \mathcal{A}$, where $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $B, D \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$.
- The initial condition is

$$y(0) = x \in \mathbb{R}^n. \tag{2}$$

Problem

Give an easily computable condition for "approximate controllability" for (1) in the general case of nonzero D with

possibly $Rank(D) < n$.

Existing Results

- Peng, S. (1994)
- Liu, Y., Peng, S. (2002)
- Buckdahn, R., Quincampoix, M., Tessitore, G. (2006)

Results due to Peng

- Consider the SDE

$$dy(t) = b(y(t), u(t))dt + \sigma(y(t), u(t))dW(t) \qquad (3)$$

Results due to Peng

- Consider the SDE

$$dy(t) = b(y(t), u(t))dt + \sigma(y(t), u(t))dW(t) \tag{3}$$

- from the initial point

$$y(0) = x \in \mathbb{R}^n, \tag{4}$$

Results due to Peng

- Consider the SDE

$$dy(t) = b(y(t), u(t))dt + \sigma(y(t), u(t))dW(t) \tag{3}$$

- from the initial point

$$y(0) = x \in \mathbb{R}^n, \tag{4}$$

- to a given terminal point

$$y(T) = \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n). \tag{5}$$

Results due to Peng

- Consider the SDE

$$dy(t) = b(y(t), u(t))dt + \sigma(y(t), u(t))dW(t) \qquad (3)$$

- from the initial point

$$y(0) = x \in \mathbb{R}^n, \qquad (4)$$

- to a given terminal point

$$y(T) = \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n). \qquad (5)$$

Definition

(3) is exactly terminal-controllable (respectively exact controllable) if, $\forall \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ (resp. $\forall \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ and $x \in \mathbb{R}^n$) $\exists u(\cdot) \in \mathcal{A}$ such that (5) (resp. (5) and (4)) holds.

The case Rank(D)=n (Peng)

$$dy(t) = (Ay(t) + Bu(t))\,dt + (Cy(t) + Du(t))\,dW(t), \qquad (1)$$

Theorem

(1) is exactly terminal-controllable if and only if

$$rank(D) = n.$$

- (1) is equivalent to

$$dy(t) = (Ay(t) + A_1u'(t) + B'u''(t))dt + u'(t)dW(t).$$

The case Rank(D)=n (Peng)

$$dy(t) = (Ay(t) + A_1 u'(t) + B' u''(t))dt + u'(t)dW(t). \qquad (6)$$

Theorem

(6) is exactly controllable if and only if

$$rank[B', AB', A_1 B', AA_1 B', A_1 AB' ...] = n.$$

The case D=0

Definitions

- (1) is approximately controllable if
  $\forall x \in \mathbb{R}^n$, $T > 0, \eta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ and $\varepsilon > 0$,
  $\exists u(\cdot) \in \mathcal{A}$ s.t.

$$E\left[\left|y(T,x,u) - \eta\right|^2\right] \leq \varepsilon$$

  (1) is approximately null controllable if the above condition

  holds for $\eta = 0$.

The case D=0

Definitions

- (1) is approximately controllable if $\forall x \in \mathbb{R}^n$, $T > 0, \eta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ and $\varepsilon > 0$, $\exists u(\cdot) \in \mathcal{A}$ s.t.

$$E\left[|y(T, x, u) - \eta|^2\right] \leq \varepsilon$$

  (1) is approximately null controllable if the above condition holds for $\eta = 0$.
- $L, M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, a linear subspace $V \subset \mathbb{R}^n$ is $(L, M)-$strictly invariant if $LV \subset Span\{V, MV\}$.

The case D=0

$$dy(t) = (Ay(t) + Bu(t))\,dt + (Cy(t) + Du(t))\,dW(t), \qquad (1)$$

Theorem

The following assertions are equivalent:

1. Equation (1) is approximately controllable.

2. Equation (1) is approximately null controllable.

3. The largest $(A^*; C^*)$-strictly invariant subspace of Ker B^* is the origin.

The general case (Rank(D)=r)

- (1) is equivalent with

$$\begin{aligned} dy(t) &= (Ay(t) + B_1 u'(t) + B_2 u''(t))dt \\ &\quad + (Cy(t) + D_1 u'(t))dW(t), \end{aligned} \tag{7}$$

where $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B_1, D_1 \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$, $B_2 \in \mathcal{L}(\mathbb{R}^{d-r}, \mathbb{R}^n)$ and $rank\ D_1 = r$.

The general case (Rank(D)=r)

- (1) is equivalent with

$$\begin{aligned} dy(t) &= (Ay(t) + B_1 u'(t) + B_2 u''(t))dt \\ &\quad + (Cy(t) + D_1 u'(t))dW(t), \end{aligned} \tag{7}$$

where $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B_1, D_1 \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$, $B_2 \in \mathcal{L}(\mathbb{R}^{d-r}, \mathbb{R}^n)$ and $rank\ D_1 = r$.

Definition

$L, M, N \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $V \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ linear subspaces. Conditioned to (N, U), V is $(L; M)$-strictly invariant if $\forall v \in V$, $\exists$ $w \in V$ s.t. $w - Nv \in U$ and $Lv + Mw \in V$.

Statement of the main result

$$dy(t) = (Ay(t) + B_1 u'(t) + B_2 u''(t))dt + (Cy(t) + D_1 u'(t))dW(t), \quad (7)$$

Theorem

We have equivalence between the following assertions:
(1) The equation (7) is approximately controllable.
(2) The equation (7) is approximately null controllable.
(3) The largest linear subspace of Ker B_2^* which, conditioned to $(F, \text{Ker } D_1^*)$ is $(A^*; C^*)$-strictly invariant is $\{0\}$ (for some F satisfying $D_1^* F + B_1^* = 0$).

The dual equation

- $$\begin{cases} dp(t) = \left[-(A^* + C^*F)p(t) - C^*q(t)\right] dt \\ \qquad\qquad +(Fp(t) + q(t))dW(t), \\ p(T) = \eta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n). \end{cases} \tag{8}$$

(where $D_1^*F + B_1^* = 0$)

The dual equation

- $$\left\{\begin{array}{l} dp(t) = \left[-(A^* + C^*F)p(t) - C^*q(t)\right] dt \\ \qquad\qquad +(Fp(t) + q(t))dW(t), \\ p(T) = \eta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n). \end{array}\right. \tag{8}$$

  (where $D_1^* F + B_1^* = 0$)
- Approximate controllability for (7) $\Longleftrightarrow$ observability condition for (8)

The dual equation

- $$\begin{cases} dp(t) = \left[-(A^* + C^*F)p(t) - C^*q(t)\right] dt \\ \qquad\qquad +(Fp(t) + q(t))dW(t), \\ p(T) = \eta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n). \end{cases} \qquad (8)$$

  (where $D_1^*F + B_1^* = 0$)
- Approximate controllability for (7) $\Longleftrightarrow$ observability condition for (8)
- $\forall T > 0,\ B_2^*p(s) = 0$ and $D_1^*q(s) = 0, P - a.s.,$ $\forall s \in [0, T] \Longrightarrow p = 0.$

FSDE

$$\begin{cases} dp(t) = \left[-(A^* + C^*F)p(t) - C^*q(t)\right] dt \\ \qquad\qquad +(Fp(t) + q(t))dW(t); \\ p(0) = \theta \in \mathbb{R}^n. \end{cases} \tag{9}$$

Lemma

(7) is approximately-controllable iff

$$\begin{aligned} \forall T &> 0,\ \theta \in \mathbb{R}^n,\ \text{and } q \in L^2_{\mathcal{P}}([0,T], Ker\ D_1^*), \\ B_2^* p(s) &= 0,\ P - a.s., \forall s \in [0,T] \\ &\Longrightarrow q(s) = 0,\ dPds - a.e \text{ and } \theta = 0 \end{aligned}$$

Conditional viability

Definition

U, $V \subset \mathbb{R}^n$ linear subspaces .

$$\begin{aligned} Viab(V/U) &= \{\theta \in V : \exists T > 0, q \in L^2_{\mathcal{P}}([0, T], U) \text{ s.t.} \\ &\quad p(s, q, \theta) \in V, P - a.s., \forall s \in [0, T]\} . \end{aligned}$$

V is conditioned to U viable locally in time (viable conditioned to U) with respect to (8) if $Viab(V/U) = V$.

Riccati equation

$$\begin{cases} P_N'(s) = -P_N(s)(A^* + C^*F) - (A + F^*C)P_N(s) + F^*P_N(s)F \\ \qquad\qquad - (F^*P_N(s) - P_N(s)C^*)(I + N\Pi_{U^\perp} + P_N(s))^{-1} \\ (P_N(s)F - CP_N(s)) + N\Pi_{V^\perp}, \\ P_N(T) = 0; \end{cases} \tag{10}$$

Theorem

$Viab(V|U) = \{\theta \in V : \ \exists T > 0 \ s.t. \ \lim_{N\to\infty} \langle P_N(T)\theta, \theta\rangle < \infty\}$

Main result

$$
\begin{aligned}
dy(t) &= (Ay(t) + B_1 u'(t) + B_2 u''(t))dt \\
&\quad +(Cy(t) + D_1 u'(t))dW(t),
\end{aligned}
\tag{7}
$$

Theorem

We have equivalence between the following assertions:
(1) The equation (7) is approximately controllable.
(2) The equation (7) is approximately null controllable.
(3) $Viab(Ker\ B_2^*|Ker\ D_1^*)$ is trivially reduced to $\{0\}$.

Characterization of conditional viability

Theorem

$V \subset \mathbb{R}^n$ viable conditioned to $U \subset \mathbb{R}^n$ iff V is $(A^*; C^*)$-strictly invariant conditioned to (F, U).

Remark

For linear subspaces $V, U \subset \mathbb{R}^n$, the largest subspace of V which is, conditionned to (N, U) $(L; M)$-strictly invariant can be obtained in at most n iterations if we set

$$\begin{aligned} V_0 &= V; \\ V_{i+1} &= \{v \in V_i : \; M((U + Nv) \cap V_i) \cap (V_i - Lv) \neq \phi\}, \\ i &\in \mathbb{N}. \end{aligned}$$

Introduction

$$\begin{cases} dX_t^{x,u} = \left(AX_t^{x,u} + Bu_t\right) dt + CX_t^{x,u} dW_t, \\ X_0 = x \in H, \end{cases} \tag{11}$$

- $\left(H, \langle \cdot, \cdot \rangle_H\right), \left(U, \langle \cdot, \cdot \rangle_U\right), \left(\Xi, \langle \cdot, \cdot \rangle_\Xi\right), \mathcal{L}(\Xi, H),\ L_2(\Xi, H)$

Introduction

$$\begin{cases} dX_t^{x,u} = \left(AX_t^{x,u} + Bu_t\right) dt + CX_t^{x,u} dW_t, \\ X_0 = x \in H, \end{cases} \tag{11}$$

- $\left(H, \langle \cdot, \cdot \rangle_H\right), \left(U, \langle \cdot, \cdot \rangle_U\right), \left(\Xi, \langle \cdot, \cdot \rangle_\Xi\right), \mathcal{L}(\Xi, H), \ L_2(\Xi, H)$
- $A : D(A) \subset H \longrightarrow H$ generates a C_0-semigroup of linear operators $\left(e^{tA}\right)_{t \geq 0}$

Introduction

$$\begin{cases} dX_t^{x,u} = \left(AX_t^{x,u} + Bu_t\right) dt + CX_t^{x,u} dW_t, \\ X_0 = x \in H, \end{cases} \tag{11}$$

- $\left(H, \langle \cdot, \cdot \rangle_H\right), \left(U, \langle \cdot, \cdot \rangle_U\right), \left(\Xi, \langle \cdot, \cdot \rangle_\Xi\right), \mathcal{L}(\Xi, H), L_2(\Xi, H)$
- $A : D(A) \subset H \longrightarrow H$ generates a C_0-semigroup of linear operators $\left(e^{tA}\right)_{t \geq 0}$
- $B \in \mathcal{L}(U, H)$, $C : H \longrightarrow \mathcal{L}(\Xi, H)$ linear (possibly unbounded)

Introduction

$$\begin{cases} dX_t^{x,u} = \left(AX_t^{x,u} + Bu_t\right) dt + CX_t^{x,u} dW_t, \\ X_0 = x \in H, \end{cases} \tag{11}$$

- $\left(H, \langle \cdot, \cdot \rangle_H\right)$, $\left(U, \langle \cdot, \cdot \rangle_U\right)$, $\left(\Xi, \langle \cdot, \cdot \rangle_\Xi\right)$, $\mathcal{L}(\Xi, H)$, $L_2(\Xi, H)$
- $A : D(A) \subset H \longrightarrow H$ generates a C_0-semigroup of linear operators $\left(e^{tA}\right)_{t \geq 0}$
- $B \in \mathcal{L}(U, H)$, $C : H \longrightarrow \mathcal{L}(\Xi, H)$ linear (possibly unbounded)
- $$\begin{aligned} &a) \; e^{tA}C \in \mathcal{L}\left(H; L_2(\Xi, H)\right), \\ &b) \; \left| e^{tA}C \right|_{\mathcal{L}(H;L_2(\Xi,H))} \leq Lt^{-\gamma}, \end{aligned}$$

  for some constants $\gamma \in \left[0, \frac{1}{2}\right)$ and $L > 0$.

Introduction

- $(\Omega, \mathcal{F}, P)$ a complete probability space

Introduction

- $(\Omega, \mathcal{F}, P)$ a complete probability space
- $(\mathcal{F}_t)_{t\geq 0}$, W a cylindrical $(\mathcal{F}_t)-$Wiener process with values in Ξ

Introduction

- $(\Omega, \mathcal{F}, P)$ a complete probability space
- $(\mathcal{F}_t)_{t\geq 0}$, W a cylindrical $(\mathcal{F}_t)-$Wiener process with values in Ξ
- $\mathcal{U}$ the space of progressively measurable $u : \mathbb{R}_+ \times \Omega \longrightarrow U$ s.t.

$$E\left[\int_0^T |u_t|^2 \, dt\right] < \infty, \ \ for \ \ all \ \ T > 0.$$

BSDE

$$\begin{cases} dY_t = -\left(A^* Y_t + C^* Z_t\right) dt + Z_t dW_t, \\ Y_T = \xi \in L^2\left(\Omega, \mathcal{F}_T, P; H\right). \end{cases} \tag{12}$$

Problem

Existence and uniqueness for the mild solution of (12)

- Assume $Ce^{tA} : H \longrightarrow L_2(\Xi; H) \Longrightarrow \left(Ce^{tA}\right)^*$ maps $L_2(\Xi; H)$ into H

- Assume $Ce^{tA} : H \longrightarrow L_2(\Xi; H) \Longrightarrow \left(Ce^{tA}\right)^* $ maps $L_2(\Xi; H)$ into H

Definition

Mild solution :(Y, Z) progressively measurable with values in H, $L_2(\Xi, H)$,
$(Y, Z) \in C\left([0, T]; L^2(\Omega; H)\right) \times L^2\left([0, T] \times \Omega; L_2(\Xi; H)\right)$,
$\sup_{t\in[0,T]} E\left[|Y_t|^2\right] + E\left[\int_0^T |Z_t|^2 dt\right] < \infty$,
$\int_0^T \left|\left(Ce^{(s-t)A}\right)^* Z_s\right| ds < \infty,\ P - a.s.$
$Y_t = e^{(T-t)A^*}\xi + \int_t^T \left(Ce^{(s-t)A}\right)^* Z_s ds - \int_t^T e^{(s-t)A^*} Z_s dW_s$,
$t \in [0, T]$.

Assumptions

A1 $C = C_1 + C_2$ s.t.

- 1. $C_2 \in \mathcal{L}(H; L_2(\Xi; H))$

A2

Assumptions

A1 $C = C_1 + C_2$ s.t.

- 1. $C_2 \in \mathcal{L}(H; L_2(\Xi; H))$
- 2. $C_1 e^{tA} \in \mathcal{L}(H; L_2(\Xi; H))$, $\forall t > 0$

$$\left| C_1 e^{tA} \right|_{\mathcal{L}(H; L_2(\Xi; H))} \leq L t^{-\gamma}, \text{ for some } \gamma \in \left[0, \frac{1}{2}\right),$$

A2

Assumptions

A1 $C = C_1 + C_2$ s.t.

- 1. $C_2 \in \mathcal{L}(H; L_2(\Xi; H))$
- 2. $C_1 e^{tA} \in \mathcal{L}(H; L_2(\Xi; H))$, $\forall t > 0$

$$\left| C_1 e^{tA} \right|_{\mathcal{L}(H;L_2(\Xi;H))} \leq L t^{-\gamma}, \text{ for some } \gamma \in \left[0, \frac{1}{2}\right),$$

- 3. $\exists a > \frac{1}{2}$ s.t.

$$A + a\left(C_1 e^{\delta A}\right)^* \left(C_1 e^{\delta A}\right) \text{ is dissipative, for some } \delta \searrow 0.$$

A2

Assumptions

A1 $C = C_1 + C_2$ s.t.

- 1. $C_2 \in \mathcal{L}(H; L_2(\Xi; H))$
- 2. $C_1 e^{tA} \in \mathcal{L}(H; L_2(\Xi; H))$, $\forall t > 0$

$$\left| C_1 e^{tA} \right|_{\mathcal{L}(H;L_2(\Xi;H))} \leq L t^{-\gamma}, \text{ for some } \gamma \in \left[0, \frac{1}{2}\right),$$

- 3. $\exists a > \frac{1}{2}$ s.t.

$$A + a \left(C_1 e^{\delta A}\right)^* \left(C_1 e^{\delta A}\right) \text{ is dissipative, for some } \delta \searrow 0.$$

A2

- $-A^2$ is dissipative.

Remark

1. If A is a self-adjoint, dissipative operator which generates a contraction semigroup, then (A2) is obviously satisfied.
2. If C_1 takes its values in $L_2(\Xi; H)$, then we may replace (A1) 3) by
3') $\exists a > \frac{1}{2}$ s.t.

$$A + aC_1^* C_1 \text{ is dissipative.}$$

Existence and uniqueness

$$\begin{cases} dY_t = -\left(A^* Y_t + C^* Z_t\right) dt + Z_t dW_t, \\ Y_T = \xi \in L^2\left(\Omega, \mathcal{F}_T, P; H\right). \end{cases} \tag{12}$$

Theorem

Under (A1) and (A2), there exists a unique mild solution of the BSDE (12). Moreover

$$\sup_{t\in[0,T]} E\left[\left|Y_t\right|^2\right] + E\left[\int_0^T \left|Z_s\right|^2 ds\right] \leq kE\left[\left|\xi\right|^2\right], \tag{13}$$

for some $k > 0$.

Proposition

(i) (11) is approximately controllable iff $\forall T > 0$*, a solution of (12) s.t.* $B^*Y_s = 0$ dP*-a.s.,* $\forall 0 \leq s \leq T$*, satisfies* $Y_s = 0$ dP*-a.s.,* $\forall 0 \leq s \leq T$*.*
(ii) (11) is approximately null-controllable iff $\forall T > 0$*, a solution of (12) s.t.* $B^*Y_s = 0$ dP*-a.s.,* $\forall 0 \leq s \leq T$*, satisfies* $Y_0 = 0$ dP*-a.e.*

Remark

If W *is 1-dimensional,* $B \in \mathcal{L}(H)$*, and* C *is a linear (possibly unbounded) operator on* H *s.t.* $A^*B^* = B^*A^*$ *and* $B^*C^* = C^*B^*$*, (11) is approximately controllable iff* $\mathcal{R}(B)$ *is dense in* H*.*

The Hautus test for deterministic linear equations

$$\begin{cases} dY_t = A^* Y_t dt, \; Y_0 = y \\ R_t = B^* Y_t = B^* e^{tA^*} y, \; t \geq 0. \end{cases}$$

$$(\mathcal{B}^* y)(t) = R_t.$$

Definition

The system (A^*, B^*) is exactly observable if $|\mathcal{B}^* y| \geq k|y|$.

If A is $n \times n$ matrix and B is $n \times 1$

Fact

(Hautus test) We have equivalence between

(1) (A^*, B^*) *is exactly observable*

(2) rank $\begin{pmatrix} A^* - sI \\ B^* \end{pmatrix} = n, \forall s \in \mathbb{C}$

(3) $|(A^* - sI)y| + |B^* y| \geq k|y|, \forall s \in \mathbb{C}, x \in \mathbb{C}^n$.

The Hautus test for deterministic linear equations

Russell, Weiss (1994)
Jacob, Partington (2006)
(JP) A is an infinitesimal generator of an exponentially stable, strongly continuous semigroup s.t. $\exists \{e_i\}, \{\lambda_i\}$,

$$Ae_i = \lambda_i e_i, \ \sup_i \lambda_i < 0.$$

$B \in \mathcal{L}(\mathbb{R}; H)$.

$$\begin{cases} dX_t^{x,u} = \left(AX_t^{x,u} + Bu_t\right) dt, \\ X_0 = x \in H, \end{cases}$$

is approximately controllable iff $\forall \ y \in H_1, \alpha < 0$,

$$|B^* y|^2 + |(A^* - \alpha I)\, y|^2 > 0 \text{ whenever } y \neq 0.$$

Approximate controllability for infinite dimension SDE

$$\begin{cases} dX_t^{x,u} = \left(AX_t^{x,u} + Bu_t\right) dt + CX_t^{x,u} dW_t, \\ X_0 = x \in H, \end{cases} \tag{11}$$

A3 A generates an exponentially stable, strongly continuous semigroup of operators.

Proposition

(necessary condition for the approximate controllability of (11))
$\forall y \in D\left(A^*\right)$, $\forall \alpha < 0$,

$$\left|B^* y\right| + \left|\left(A^* - \alpha I\right) y\right| > 0 \text{ whenever } y \neq 0. \tag{N1}$$

Approximate controllability for infinite dimension SDE

$$\begin{cases} dX_t^{x,u} = \left(AX_t^{x,u} + Bu_t\right) dt + CX_t^{x,u} dW_t, \\ X_0 = x \in H, \end{cases} \tag{11}$$

- W is 1-dimensional,
- $C \in \mathcal{L}(H)$,
- U is a bounded closed subspace of some separable real Hilbert space V,
- $B \in \mathcal{L}\left(V; H\right)$.

Proposition

(necessary condition for the approximate controllability of (11)
$\forall y \in D\left(A^*\right)$, $\forall \alpha < 0$,

$$|B^* y| + |(A^* + \lambda C^* - \alpha I) y| > 0, \ \forall y \neq 0, \ (\lambda, \alpha) \in \mathbb{R} \times \mathbb{R}_-. \tag{N2}$$

Example

$$\begin{cases} d_t X^u(t,x) = \sum_{i,j=1}^N \partial_i \left(a_{i,j}(x)\partial_j X^u(t,x)\right) dt + u(t)\, b(x) dt \\ \qquad\qquad + \sum_{i=1}^N c_i(x)\partial_i X^u(t,x) dW_t, \\ X^u(t,x) = 0,\ \forall (t,x) \in [0,T] \times \partial\mathcal{O}, \\ X^u(0,x) = \xi(x),\ \forall x \in \mathcal{O}, \end{cases} \tag{14}$$

- $\mathcal{O} \subset \mathbb{R}^N$ is a regular domain,
- $a(x) = \sigma(x)\sigma^*(x)$
- $c = (c_1, \ldots, c_N) \in C^\infty\left(\mathcal{O}; \mathbb{R}^N\right)$, $b \in H^1(\mathcal{O})$
- $\sum_{i,j=1}^N \left(a_{i,j}(x) - \alpha c_i(x)c_j(x)\right)\lambda_i\lambda_j \geq 0$, for some $\alpha > \frac{1}{2}$

Example

$$\begin{cases} d_t Y(t,x) = -\left(\sum_{i,j=1}^{N} \partial_i \left(a_{i,j}(x)\partial_j Y\left(t,x\right)\right)\right) dt \\ + \left(\sum_{i=1}^{N} c_i(x)\partial_i Z(t,x)\right) dt \\ + \left(\sum_{i=1}^{N} \partial_i c_i(x) Z(t,x)\right) dt + Z\left(t,x\right) dW_t, \\ Y(t,x) = Z(t,x) = 0, \ \forall \left(t,x\right) \in [0,T] \times \partial\mathcal{O}, \\ Y(T,x) = \eta(x), \ \forall x \in \mathcal{O}, \end{cases}$$

if (14) is approximately controllable and $\zeta_n(x)$ is a complete orthonormal base of eigenvectors for A, then every coefficient of b in this base must be non null.

References

- Buckdahn, R., Quincampoix, M., Tessitore, G., *A Characterization of Approximately Controllable Linear Stochastic Differential Equations,* Stochastic Partial Differential Equations and Applications, G. Da Prato and L. Tubaro Eds Series of Lecture Notes in Pure and Appl. Math., Chapman & Hall Vol.245 (2006), pp. 253-260
- Liu, Y., Peng, S., *Infinite horizon backward stochastic differential equation and exponential convergence index assignment of stochastic control systems*, Automatica, 38 (2002), pp. 1417-1423.
- Pardoux, E., Peng, S.G., *Adapted solutions of a backward stochastic differential equation*, Systems and Control Letters, 14 (1990), pp. 55-61.
- Peng, S.G., *Backward Stochastic Differential Equation and*

Exact Controllability of Stochastic Control Systems, Progr. Natur. Sci. vol. 4, No. 3 (1994), pp. 274-284.

- Confortola, F., *Dissipative backward stochastic differential equations in infinite dimensions,* 2004.
- Jacob, B., Partington, J., R., *On controllability of diagonal systems with one-dimensional input space,* Systems and Control Letters 55 (2006), pp. 321 – 328.
- Russell, D.L., Weiss, G, *A general necessary condition for exact observability*, SIAM J. Control Optim. 32 (1) (1994), pp. 1–23.
- Sirbu, M, Tessitore, G., *Null controllability of an infinite dimensional SDE with state and control-dependent noise*, Systems and Control Letters, 44 (2001), pp. 385-394.

Thank you for your attention!