On the Backward Stochastic Riccati Equation in Infinite dimensions

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PLAN

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Abstract setting

- Let H, U and Ξ be real separable Hilbert spaces, endowed respectively with the norms $|\cdot|_H$ and $|\cdot|_\Xi$.

- Let W be a cylindrical Wiener process defined on a complete probability basis $(\Omega, \mathcal{F}, \mathbb{P})$ with value in Ξ. We denote by $\mathcal{F}_t$ for $t \geq 0$ its natural filtration completed.

- Let $A : D(A) \subset H \to H$ be an unbounded operator that generates a C_0-semigroup.

We consider a quadratic optimal control problem for a system governed by the following *state equation* for $0 \leq t \leq T$:

$$\begin{cases} dy(t) = (Ay(t) + B(t)u(t))\,dt + C(t)y(t)\,dW(t), \\ \\ y(0) = x \end{cases} \tag{1}$$

where $y \in H$ is the **state** of the system and $u \in U$ is the **control**, B and C are allowed to be predictable processes with values in suitable spaces of operators.

Our purpose is to minimize over all admissible controls u the quadratic cost functional:

$$J(0, x, u) = \mathbb{E}\int_0^T \left(|\sqrt{S}(s)y(s)|_H^2 + |u(s)|_H^2\right) ds + \mathbb{E}\langle P_T y(T), y(T)\rangle_H \tag{2}$$

where S may be a predictable process and P_T may be an $\mathcal{F}_T$- measurable random variable both with values in suitable spaces of operators.

The controlled heat equation

$$\begin{cases} d_t y(t,\xi) = \Delta_\xi y(t,\xi)dt + b(t,\xi)u(t,\xi)dt + \sum_{i=1}^{\infty} c_i(t,\xi)y(t,\xi)d\beta_i(t), (t,\xi) \in [0,T]\times\mathcal{D}, \\ y(t,\xi) = 0, \quad \xi \in \partial\mathcal{D},\ t \in [0,T], \\ y(0,\xi) = x(\xi), \qquad \xi \in \mathcal{D},\ t \in [0,T]. \end{cases} \tag{3}$$

$$J = \mathbb{E}\left[\int_0^T \int_{\mathcal{D}} \zeta(t,\xi)y^2(t,\xi)d\xi\, dt + \int_{\mathcal{D}} \pi(\xi)y^2(T,\xi)d\xi\right]$$

Wave equation in random media with stochastic damping

$$\begin{cases} d_t\partial_t\xi(t,\zeta) = \Delta_\zeta\xi(t,\zeta)dt + b(t,\zeta)u(t,\zeta)dt + \mu(t,\zeta)\partial_t\xi(t,\zeta)dt + \sum_{i=1}^{\infty} c_i(t,\zeta)\xi(t,\zeta)d\beta_i(t), \\ \xi(t,\zeta) = 0, \qquad \zeta\in\partial\mathcal{D},\ t\in[0,+\infty), \\ \xi(0,\zeta) = x_0(\zeta),\ \partial_t\xi(0,\zeta) = v_0(\zeta) \qquad \zeta \in \mathcal{D}, \end{cases} \tag{4}$$

$$J = \mathbb{E}\int_0^{+\infty}\int_{\mathcal{D}} \left[u^2(t,\zeta)d\zeta + \kappa_1(t,\zeta)(\nabla_x\xi(t,\zeta))^2 + \kappa_2(t,\zeta)\left(\frac{\partial\xi}{\partial t}(t,\zeta)\right)^2\right] d\zeta\, dt$$

Assume for the moment that the coefficients A, B, C and the data S, P_T are all deterministic.

Due to the quadratic *nature* of the cost functional and the linearity of the state equation the value function V takes the form

$$V(t,x) = \langle P(t)x, x\rangle$$

where for P is an operator valued function. So in this case the Hamilton-Jacobi- Bellman equation "reduces" to an ordinary differential equation with value in $\Sigma(H)$ (the space of symmetric linear and bounded operators from H to H):

$$\begin{cases} -dP(t) & = (A^*P(t) + P(t)A + \mathrm{Tr}[C^*(t)P(t)C(t)])\,dt \\ & \quad -(P(t)B(t)B^*(t)P(t) - S(t))\,dt, \quad t \in [0,T] \\ P(T) & = P_T. \end{cases} \tag{5}$$

That is the well known Riccati equation - for stochastic linear quadratic games- see **Wonham['68 finite dimensions]** or **Ichikawa ['76–'84 infinite dimensions]** and bibliography therein.

In the infinite dimensional case being A in general an unbounded operator, equation (5) has to be understood in the following *mild* sense:

$$
\begin{aligned}
P(t) = e^{A^*(T-t)} P_T e^{A(T-t)} &+ \int_t^T e^{A^*(s-t)} S(s) e^{A(s-t)}\, ds \\
&+ \int_t^T e^{A^*(s-t)} \mathrm{Tr}[C^*(s)P(s)C(s)] e^{A(s-t)}\, ds \\
&- \int_t^T e^{A^*(s-t)} P(s)B(s)B^*(s)P(s) e^{A(s-t)}\, ds
\end{aligned}
$$

($e^{tA} \in L(H)$ for every $t \geq 0$)

Now we allow the coefficients and the data to be random

Analogously to the deterministic case we define the "stochastic" value function

$$\langle P(t)x,x\rangle_H \doteq \inf_u \mathbb{E}^{\mathcal{F}_t}[\int_t^T \left(|\sqrt{S}(s)y(s)|_H^2 + |u(s)|_H^2\right) ds + \langle P_T y(T), y(T)\rangle_H] \quad (6)$$

Notice that P is an adapted stochastic process that formally verifies the following backward stochastic differential equation (B.S.R.E.):

$$\begin{cases} -dP(t) = (A^*P(t) + P(t)A + \mathrm{Tr}[C^*(t)P(t)C(t)])\,dt \\ +\mathrm{Tr}[C^*(t)Q(t) + Q(t)C(t)]\,dt - P(t)B(t)B^*(t)P(t)\,dt \\ +S(t)\,dt + Q(t)\,dW(t), \qquad t \in [0,T], \\ P(T) = P_T. \end{cases} \quad (7)$$

where the unknown is the couple (P,Q).

Backward Stochastic Riccati Equations

The theory of Backward Stochastic Riccati Equations in finite dimension (and finite horizon) is well developed. Here there is the most general formulation:

$$\begin{cases} -dK = [(A^*K + KA + C_i^*KC_i + C_i^*L_i + L_iC_i]) \, dt \\ -(KB + (C_i^*L_iD_i + L_iD_i))(N + D_i^*KD_i)^{-1}(KB + (C_i^*L_iD_i + L_iD_i))^* \, dt \\ -S \, dt + L_i \, dW_i, \qquad t \in [0, T], \\ P(T) = P_T. \end{cases} \tag{8}$$

where $W(t) = (W_1(t), \cdots, W_d(t))$ is a d-dimensional Brownian motion, the coefficients A, B, C_i and D_i are $\{\mathcal{F}_t\}$-progressively measurable bounded matrix-valued processes and M is an $\mathcal{F}_T$-measurable nonnegative bounded random matrix while Q and N are $\{\mathcal{F}_t\}$-progressively measurable nonnegative and bounded matrix-valued processes.

The previous equation arise from the solution of the optimal control problem:

$$\inf_{u \in L^2_{\mathcal{P}}(0,T;\mathbb{R}^m)} J(u;0,x)$$

where for $t \in [0,T]$ and $x \in \mathbb{R}^m$,

$$J(u;t,x) = \mathbb{E}^{\mathcal{F}_t}\{[(My^{t,x,u}(T), y^{t,x,u}(T))$$
$$+ \mathbb{E}^{\mathcal{F}_t}\{\int_t^T (N(s)u(s), u(s)) + (Q(s)y^{t,x,u}(s), y^{t,x,u}(s))]\, ds$$

and $y^{t,x,u}$ is the solution to the following stochastic differential equation:

$$\begin{cases} dy = (Ay + Bu)\, ds + \sum_{i=1}^d (C_i y + D_i u)\, dW_i \quad t \le s \le T, \\ y(t) = x \end{cases}$$

These equations and its relations with the theory of the linear quadratic optimal stochastic control have been the object of several studies.

- **Bismut [1976-78]**: he pointed out the difficulty in treating the non linearity -in the two unknown variables- of the drift term.
- **Peng [1998]** included in his list of open problem the well posedeness of the general BRSDE.
- **Peng [1992]** (complete the case with $D = 0$), then **Kohlmann and Zhou [2000]**, **Kohlmann and Tang [2001-2002]**, **Tang [2003]**(general case $D \neq 0$).

The infinite dimensional case

- The main difficulty in the infinite dimensional case is that equation (8) has value in the space of symmetric linear and bounded operators from H into H ($:=\Sigma(H)$) that **is not a Hilbert space**.

- We start our analysis of the Riccati equation in the Hilbert space of Hilbert-Schmidt, symmetric, linear and bounded operators from H into H ($:=\Sigma_2(H)$) and then we extend our result in a more general framework.

- As in **S.Peng [‘92]** we consider a simplified version of the Backward Stochastic Riccati equation setting $D_i = 0$ for all $i = 1, \dots, d$ and $N = I$.

This is the Backward Stochastic Riccati Equation (BRSE) we are going to study:

$$\begin{cases} -dP(t) = (A^*P(t) + P(t)A + \mathrm{Tr}[C^*(t)P(t)C(t)])\,dt \\ +\mathrm{Tr}[C^*(t)Q(t) + Q(t)C(t)]\,dt - P(t)B(t)B^*(t)P(t)\,dt \\ +S(t)\,dt + Q(t)\,dW(t), \qquad t \in [0,T], \\ P(T) = P_T. \end{cases} \tag{9}$$

The operator C is of the form: $C = \sum_{i=1}^{\infty} C_i(\cdot, f_i)_\Xi$, where $\{f_i : i \in \mathbb{N}\}$ is an orthonormal basis in Ξ. Therefore it has to be intended:

$$\mathrm{Tr}[C^*(t)P(t)C(t) + C^*(t)Q(t) + Q(t)C(t)] = \sum_{i=1}^{+\infty}[C_i^*(t)P(t)C_i(t) + C_i^*(t)Q_i(t) + Q_i(t)C_i(t)]$$

We will first consider this equation with values in the Hilbert space $\Sigma_2(H)$ of symmetric, non-negative and Hilbert Schmidt operators, then with values in $\Sigma(H)$.

Hypotheses

A1) $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0 semigroup $e^{tA} : H \to H$.
We denote by M_A a positive constant such that:

$$\sup_{t\in[0,T]} |e^{tA}|_{L(H)} \leq M_A$$

A2) We assume that $B \in L^{\infty}_{\mathcal{P},S}((0,T)\times\Omega; L(U,H))$. We denote by M_B a positive constant such that:

$$|B(t,\omega)|_{L(U,H)} < M_B, \quad \mathbb{P} - \text{a.s. and for a.e. } t \in (0,T).$$

A3) We assume that C is of the form: $C = \sum_{i=1}^{\infty} C_i(\cdot, f_i)_\Xi$, where $\{f_i : i \in \mathbb{N}\}$ is an orthonormal basis in Ξ. Moreover we suppose that

$$C_i \in L^{\infty}_{\mathcal{P},S}((0,T) \times \Omega; L(H))$$

and

$$\left(\sum_{i=1}^{\infty} |C_i(t,\omega)|^2_{L(H)} \right)^{1/2} < M_C,$$

$\mathbb{P}-$a.s. for a.e. $t \in (0,T)$ for a suitable positive constant M_C.

A4) $S \in L^1_{\mathcal{P},S}((0,T); L^\infty(\Omega; \Sigma^+(H)))$, $P_T \in L^\infty_S(\Omega, \mathcal{F}_T; \Sigma^+(H))$.

A5) $S \in L^2_{\mathcal{P}}(\Omega \times (0,T); \Sigma_2^+(H))$, $P_T \in L^2(\Omega, \mathcal{F}_T; \Sigma_2^+(H))$.

BSRE in the Hilbert Space $\Sigma_2(H)$

Definition 1 *Fix* $T_0 \in [0,T]$. *A* mild *solution for problem (9), considered in* $[T_0,T]$ *is a pair* (P,Q) *with*

$$P \in L^2_{\mathcal{P}}(\Omega, C([T_0,T];\Sigma_2(H))) \cap L^{\infty}_{\mathcal{P},S}(\Omega; C([T_0,T];\Sigma^+(H)))$$

$$Q \in L^2_{\mathcal{P}}(\Omega \times [T_0,T]; L_2(\Xi;\Sigma_2(H)))$$

such that for all $t \in [T_0,T]$*:*

$$\begin{aligned} P(t) &= e^{(T-t)A^*} P_T e^{(T-t)A} + \int_t^T e^{(s-t)A^*} S(s) e^{(s-t)A}\, ds + \\ &\int_t^T e^{(s-t)A^*} \mathrm{Tr}\Big[C^*(s)P(s)C(s) + C^*(s)Q(s) + Q(s)C(s)\Big] e^{(s-t)A}\, ds \\ &- \int_t^T e^{(s-t)A^*} P(s)B(s)B^*(s)P(s) e^{(s-t)A}\, ds \\ &+ \int_t^T e^{(s-t)A^*} Q(s) e^{(s-t)A}\, dW(s) \qquad \mathbb{P}-a.s. \end{aligned} \tag{10}$$

Theorem 1 *Assume A1)-A5). Problem* (9) *has a unique mild solution* (P, Q).

- Difficulties:
  a) unboundedness of A;

  b) nonlinearity of quadratic type in the P variable.

- Solutions:
  a) Introduction of Yoshida approximants to apply Itô formula and then pass to the limit;

  b) 1) local existence and uniqueness obtained by fixed point technique (some tools from **Hu and Peng [1991]**);
  2) existence in large by a-priori estimates on the $L(H)$-norm of P arising from control interpretation.

Finite Horizon case: synthesis of the optimal control

Theorem 2 *Fix $T > 0$ and $x \in H$. Then:*

1. *There exists a unique* optimal control*. That is a unique control $\bar{u} \in L^2_{\mathcal{P}}(\Omega \times [0,T]; U)$ such that:*

$$J(0, x, \bar{u}) = \inf_{u \in L^2_{\mathcal{P}}(\Omega \times [0,T];U)} J(0, x, u)$$

2. *If $\bar{y}$ is the mild solution of the state equation corresponding to $\bar{u}$ (that is the* optimal state*) then $\bar{y}$ satisfies the* closed loop *equation*

$$\begin{cases} d\bar{y}(t) = [A\bar{y}(t) - B(t)B(t)^* P(t)\bar{y}(t)]\, dt + \; C(t)\bar{y}(t)\, dW(t), \qquad 0 \le t \le T \\ \\ \bar{y}(0) = x \end{cases} \tag{11}$$

3. *The following feedback law holds $\mathbb{P}$-a.s. for almost every s.*

$$\bar{u}(s) = -B^*(s)P(s)\bar{y}(s). \tag{12}$$

4. *The optimal cost is given by $J(0, x, \bar{u}) = \langle P(0)x, x\rangle_H$, for all $x \in H$.*

Key point: The fundamental relation holds:

$\forall t \in [0,T]$, $\forall x \in H$ and $\forall u \in L^2_{\mathcal{P}}(\Omega \times [0,T]; U)$:

$$\langle P(t)x, x\rangle_H = J(t,x,u) - \mathbb{E}^{\mathcal{F}_t} \int_t^T |u(s) + B^*(s)P(s)y(s)|^2 \, ds, \ \mathbb{P} - \text{a.s.}$$

BSRE in the Banach Space $\Sigma(H)$

In what follows we drop hypothesis A5).

Definition 2 *A process* $P \in L^{\infty}_{\mathcal{P},S}(\Omega \times (0,T); \Sigma^{+}(H))$, *is a* generalized solution *if there exists a sequence* (S^N, P^N, Q^N) *where:*

(i) $S^N \in L^1_{\mathcal{P},S}((0,T); L^{\infty}(\Omega; \Sigma^{+}(H)) \cap L^2_{\mathcal{P}}(\Omega \times (0,T); \Sigma_2(H))$ *and there exists a positive function* $c \in L^1([0,T])$ *such that* $|S^N(t)|_{L(H)} \leq c(t)$, *for all* $N \in \mathbb{N}$, $\mathbb{P}$*-a.s. for a.e.* $t \in [0,T]$.

(ii) the pair (P^N, Q^N) *is the unique mild solution of the following B.R.S.E.:*

$$\begin{cases} -dP^N(t) = (A^*P^N(t) + P^N(t)A + \mathrm{Tr}[C^*(t)P^N(t)C(t)])\,dt \\ (\mathrm{Tr}[C^*(t)Q^N(t) + Q^N(t)C(t)] - P^N(t)B(t)B^*(t)P^N(t))\,dt \\ S^N(t)\,dt + Q^N(t)\,dW(t),\ t \in [0,T] \\ P^N(T) = P^N_T \end{cases}$$

in the space of H.S. operators.

(iii) for all $x \in H$*:*

$$S^N(t,\omega)x \to S(t,\omega)x \text{ in } H \quad \mathbb{P} \text{ a.s. for a.e. } t \in [0,T]$$

(iv) for every $t \in [0,T]$ *and for all* $x \in H$*:*

$$P^N(t,\omega)x \to P(t,\omega)x \quad \text{ in } H \qquad \mathbb{P} \text{ a.s.}$$

Theorem 3 *Assume that hypotheses A1)-A4) hold true.*
Then there exists a unique generalized solution of problem (9). Moreover we have the following characterization of the optimal control: fix $T > 0$ *and* $x \in H$*, then:*

1. *there exists a unique control* $\overline{u} \in L^2_{\mathcal{P}}(\Omega \times [0,T]; U)$ *such that:*
$$J(0,x,\overline{u}) = \inf_{u \in L^2_{\mathcal{P}}(\Omega \times [0,T];U)} J(0,x,u)$$

2. *If* $\overline{y}$ *is the mild solution of the state equation corresponding to* $\overline{u}$ *(that is the optimal state), then* $\overline{y}$ *is the unique mild solution to the* closed loop *equation:*
$$\begin{cases} d\overline{y}(t) = [A\overline{y}(t) - B(t)B^*(t)P(t)\overline{y}(t)]\,dt + C\overline{y}(t)\,dW(t) \\ \overline{y}(0) = x \end{cases} \tag{13}$$

3. *The following feedback law holds* $\mathbb{P}$*-a.s., for a.e.* $s \in [0,T]$*:*
$$\overline{u}(s) = -B^*(s)P(s)\overline{y}(s).$$

4. *The optimal cost is given by* $J(0,x,\overline{u}) = \langle P(0)x, x\rangle_H$*.*

Finite Horizon case: synthesis of the optimal control

Proof (idea)

1. Existence: Finite dimensional projections.

2. Uniqueness and synthesis of the optimal control: control dependent interpretation.

An example: the controlled heat equation

$$\begin{cases} d_t y(t,\xi) = \Delta_\xi y(t,\xi)dt + b(t,\xi)u(t,\xi)dt + \sum_{i=1}^{\infty} c_i(t,\xi)y(t,\xi)d\beta_i(t), (t,\xi) \in [0,T] \times \mathcal{D}, \\ y(t,\xi) = 0, \qquad \xi \in \partial\mathcal{D},\ t \in [0,T], \\ y(0,\xi) = x(\xi)\ \ (t,\xi) \in [0,T] \times \mathcal{D}. \end{cases} \tag{14}$$

and the cost functional

$$J(0,x,u) = \mathbb{E}[\int_0^T \int_{\mathcal{D}} \zeta(t,\xi)y^2(t,\xi)d\xi\, dt + \int_{\mathcal{D}} \pi(\xi)y^2(T,\xi)d\xi]$$

In the above formulae $\mathcal{D} \subset \mathbb{R}^d$ is a bounded domain with regular boundary. By $\mathcal{B}(\mathcal{D})$ we denote the Borel σ-field in $\mathcal{D}$ and λ is the Lebesgue measure.

Moreover $\{\beta_i : i = 1, 2...\}$ are independent standard (real valued) brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We set $\mathcal{F}_t = \sigma\{\beta_i(s) : s \in [0,t],\ i = 1, 2...\}$ and denote by $\mathcal{P}$ the predictable σ-field in $\Omega \times [0,T]$.

On the coefficients we assume the following:

1. b, ζ, and c_i, $i = 1, 2, ..$ are measurable maps from $(([0,T] \times \Omega) \times \mathcal{D})$ endowed with the σ-field $\mathcal{P} \otimes \mathcal{B}(\mathcal{D})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

2. there exists a constant $K > 0$ such that:
$$|b(t,\xi)| + \sum_{i=1}^{\infty} |c_i(t,\xi)|^2 \leq K, \ \mathbb{P} - \text{a.s., for a.e.} (t,\xi) \in [0,T] \times \mathcal{D}$$

3. ζ has values in $\mathbb{R}^+$ and there exists $k \in L^1([0,T];\mathbb{R})$ such that
$$\zeta(t,\xi) \leq k(t), \qquad \mathbb{P} - \text{a.s., for a.e.} (t,\xi) \in [0,T] \times \mathcal{D}$$

4. $\pi \in L^\infty(\Omega \times \mathcal{D}, \mathcal{F}_T \otimes \mathcal{B}(\mathcal{D}), \mathbb{P} \otimes \lambda; \mathbb{R}^+)$

To fit our abstract setting we let:

1. $H = U = L^2(\mathcal{D})$
2. Ξ is any separable real Hilbert space with orthonormal basis $\{f_i : i = 1, 2, ...\}$ and $W(t) = \sum_{i=1}^{\infty} f_i \beta_i(t)$
3. $\mathcal{D}(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, $A\phi = \Delta\phi$ for all $\phi \in \mathcal{D}(A)$
4. $(B(t)\phi)(\xi) = b(t, \xi)\phi(\xi)$, $(C_i(t)\phi)(\xi) = c_i(t, \xi)\phi(\xi)$, $(S(t)\phi)(\xi) = \zeta(t, \xi)\phi(\xi)$, $(P_T\phi)(\xi) = \pi(\xi)\phi(\xi)$

Theorem 4 *The assumptions of Theorem 3 hold.*

Remark 5 *Under general assumptions on* ζ *and* π, S *and* P_T *verify assumption A4). Assumption A5) would not be verified even considering* $\zeta \equiv 1$, $\pi \equiv 1$.

Remark 6 *If* b *takes only values in* $\{0, 1\}$ *and we set* $\mathcal{O}(t, \omega) = \{\xi : b(t, \omega, \xi) = 1\}$ *the above control problem where the stochastic heat equation can be controlled only in the subdomain* $\mathcal{O} \subset \mathcal{D}$.

Remark 7 *Notice that none on the special regularity features on the heat semigroup have been used here.*

Infinite Horizon case

Now we consider the state equation in the whole positive time line:

$$\begin{cases} dy(s) = (Ay(s) + B(s)u(s))\, ds + \ C(s)y(s)\, dW(s) \quad s \geq 0 \\ \\ y(0) = x \end{cases} \tag{15}$$

We want to minimize over all admissible controls u the quadratic cost functional:

$$J_\infty(0, x, u) = \mathbb{E} \int_0^{+\infty} \left(|\sqrt{S}(s)y(s)|_H^2 + |u(s)|_H^2 \right) ds \tag{16}$$

where S is a predictable process with values in suitable spaces of operators. Let us define the "stochastic" value function

$$\langle P(t)x, x\rangle_H \doteq \inf_u \mathbb{E}^{\mathcal{F}_t} \left[\int_t^{+\infty} \left(|\sqrt{S}(s)y(s)|_H^2 + |u(s)|_H^2 \right) ds \right] \tag{17}$$

Hypotheses: the same requirements of the finite horizon case extended on $(0, +\infty)$:

A2) We assume that $B \in L^{\infty}_{\mathcal{P},S}((0, +\infty) \times \Omega; L(U, H))$.

A3) As before $C = \sum_{i=1}^{\infty} C_i(\cdot, f_i)_{\Xi}$ and now we suppose that

$$C_i \in L^{\infty}_{\mathcal{P},S}((0, +\infty) \times \Omega; L(H)).$$

A4) $S \in L^{1}_{\mathcal{P},S}((0, +\infty); L^{\infty}(\Omega; \Sigma^{+}(H)))$.

.

The Backward Stochastic Riccati Equation corresponding to this problem is hence defined in the whole $[0, +\infty)$:

$$\left\{ \begin{array}{l} -dP(t) = (A^*P(t)+P(t)A+A_\sharp^*(t)P(t) + P(t)A_\sharp(t))\,dt \\ (S(t) + \mathrm{Tr}[C^*(t)P(t)C(t) + C^*(t)Q(t) + Q(t)C(t)])\,dt \\ -P(t)B(t)B^*(t)P(t)\,dt + Q(t)\,dW(t),\ t \geq 0 \end{array} \right. \tag{18}$$

where the final condition has disappeared.

Definition 3 *We say that P is a solution to (18) if it is an $\{\mathcal{F}_t\}_{t\geq 0}$- adapted process that takes values in $\Sigma^+(H)$ such that for every fixed $T > 0$ it is a generalized solution of the Riccati equation (7) in $[0, T]$ with final data $P_T = P(T)$.*

If (A, B, C) and S are deterministic and autonomous (independent of t) the Riccati equation corresponding to the infinite horizon problem is the following algebraic equation

$$A^*X + XA + \sum_{i=1}^{+\infty} C_i^* X C_i - XBB^*X + S = 0$$

this equation admits a minimal solution whenever the following condition holds:

$$\forall x \in H, \exists u \in L^2_{\mathcal{P}}((0, +\infty) \times \Omega; H)$$

$$\mathbb{E} \int_0^{+\infty} (|\sqrt{S}(t) X^{x,u}(t)|^2 + |u(t)|^2)\, dt < +\infty$$

(finite cost condition or stabilizability of the coefficients relatively to $\sqrt{S}$.)

In the general case, being the coefficients stochastic processes, the condition that garanties the existence of a solution turns out to be the following:

Definition 4 *We say that* (A, B, C) *is stabilizable relatively to the observations* $\sqrt{S}$ *if, for all* $t \in [0, +\infty)$ *and all* $x \in H$ *there exists a control* $u \in L^2_{\mathcal{P}}([t, +\infty) \times \Omega; U)$ *such that*

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} (|\sqrt{S}(s) y^{t,x,u}(s)|^2 + |u(s)|^2)\, ds < K \qquad \mathbb{P} - a.s. \tag{19}$$

for some positive constant K *(that may depend on* t *and* x*).*

We have the following:

Proposition 8 *Assume* $A1) - A4)$ *and that* (A, B, C) *is stabilizable relatively to the observations* $\sqrt{S}$ *, then there exists a solution of the Riccati equation* (18) *in* $[0, +\infty)$*.*

A solution $\bar{P}$ of equation (18) is obtained as a pointwise limit of the sequence $\{P^N\}_{N \in \mathbb{N}}$ of generalized solutions of the Riccati equation (7) in $[0, N]$ with final data $P^N(N) = 0$.

$\bar{P}$ is the minimal solution and is not in general uniformly bounded in t.

Theorem 9 *Assume $A1) - -A4)$ and that (A, B, C) is stabilizable relatively to $\sqrt{S}$. Fix $x \in H$, then:*

1. *there exists a unique control $\overline{u} \in L^2_{\mathcal{P}}(\Omega \times [0, +\infty); U)$ such that:*

$$J_\infty(0, x, \overline{u}) = \inf_{u \in L^2_{\mathcal{P}}(\Omega \times [0,+\infty); U)} J_\infty(0, x, u)$$

2. *If $\overline{y}$ is the mild solution of the state equation corresponding to $\overline{u}$ (that is the optimal state), then $\overline{y}$ is the unique mild solution to the* closed loop *equation:*

$$\begin{cases} d\overline{y}(r) = [A\overline{y}(r) - B(r)B^*(r)\bar{P}(r)\overline{y}(r)]\, dr + C\overline{y}(r)\, dW(r),\ t > 0 \\ \overline{y}(0) = x \end{cases} \tag{20}$$

3. *The following feedback law holds $\mathbb{P}$-a.s. for almost every s.*

$$\overline{u}(s) = -B^*(s)\bar{P}(s)\overline{y}(s). \tag{21}$$

4. *The optimal cost is given by $J_\infty(0, x, \overline{u}) = \langle \bar{P}(0)x, x \rangle_H$.*

Assume from now on that (A, B, C) is stabilizable relatively to $\sqrt{S}$.

We introduce the following definition.

Definition 5 *Let P be a solution to* (18). *We say that P stabilize (A, B, C) relatively to I* uniformly in time *if for every $t > 0$ and $x \in H$ there exists a positive constant M, independent of t, such that*

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} |y^{t,x}(r)|_H^2 \, dr \le M \qquad \mathbb{P} - a.s. \tag{22}$$

where $y^{t,x}$ is the mild solution to:

$$\begin{cases} dy^{t,x}(s) = [A - B(s)B^*(s)P(s)]y^{t,x}(s)\, ds + C(s)y^{t,x}(s)\, dW(s), \quad s \ge t \\ y^{t,x}(t) = x \end{cases} \tag{23}$$

Proposition 10 *Assume that $\bar{P}$ stabilize the coefficients (A, B, C) relatively to I and it is bounded in time, i.e. for every $t \geq 0$ there exists a constant $M > 0$ such that*

$$|\bar{P}(t)|_{L(H)} \leq M \qquad\qquad \mathbb{P} - a.s.$$

Then $\bar{P}$ is the unique solution of equation (18) among the solutions that are uniformly bounded.

Remark: If the constant K in (19) is independent of t then the minimal solution $\bar{P}$ is uniformly bounded, that is if for every $t \geq 0$ there exists a constant $M > 0$ such that

$$|\bar{P}(t)|_{L(H)} \leq M \qquad\qquad \mathbb{P} - a.s.$$

Remark: $S \geq \beta I$ + boundedness in time imply that $\bar{P}$ stabilize (A, B, C) relatively to I.

Let

1. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis such that $\mathcal{F}_0$ contains the $\mathbb{P}$ null sets of $\mathcal{F}$;

2. $W_t : t \geq 0$ be a cylindrical Wiener process with values in Ξ that is independent of $\mathcal{F}_0$ and $\mathcal{F}_t := \mathcal{F}_0 \cup \sigma(W_s : 0 \leq s \leq t)$;

3. $\theta_t : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ measurable function such that:

   (a) $\theta_0 =$ id and $\theta_{t+s} = \theta_t \cdot \theta_s$ for all $t, s \geq 0$;

   (b) $\mathbb{P} \cdot \theta_t^{-1} = \mathbb{P},\ t \geq 0$;

   (c) $\theta_t^{-1}(\mathcal{F}_0) = \mathcal{F}_t,\ t \geq 0$;

   (d) $W_s \cdot \theta_t = W_{s+t} - W_t$

Definition 6 *A process $X : [0, +\infty) \times \Omega \to E$, where E is a generic Banach space, is stationary if for all $t, s \geq 0$*

$$X_t \cdot \theta_s = X_{t+s} \qquad \mathbb{P} - a.s.$$

Assume that B, C, S are stationary processes, then we have the following

Proposition 11 *If (A, B, C) is $\sqrt{S}$ stabilizable, then the minimal solution $\overline{P}$ of the infinite horizon stochastic Riccati equation (18) is stationary.*

Remark: Notice that in the stationary framework it is enough to check stabilizability of the coefficients in $t = 0$.

Indeed if

$$\mathbb{E}^{\mathcal{F}_0} \int_0^{+\infty} (|\sqrt{S}(s) y^{0,x,u}(s)|^2 + |u(s)|^2)\, ds < K \qquad \mathbb{P} - a.s. \tag{24}$$

then there exists a constant M such that for all $\rho > 0$:

$$|P^{\rho}(0)|_{L(H)} \leq M \qquad\qquad \mathbb{P} - \text{a.s.}$$

but since $P^{\rho}(t) = P^{\rho - t}(0) \cdot \theta_t$ then for all $t \geq 0$ and all $\rho > t$:

$$|P^{\rho}(t)|_{L(H)} \leq M \qquad\qquad \mathbb{P} - \text{a.s.}$$

and this condition implies the stabilizability of the coefficients with respect to $\sqrt{S}$.

Optimal control for a wave equation in random media with stochastic damping

$$\begin{cases} d_t\partial_t\xi(t,\zeta) = \Delta_\zeta\xi(t,\zeta)dt + b(t,\zeta)u(t,\zeta)dt + \mu(t,\zeta)\partial_t\xi(t,\zeta)dt \\ \qquad\qquad + \sum_{i=1}^{\infty} c_i(t,\zeta)\xi(t,\zeta)d\beta_i(t), \qquad \zeta \in \mathcal{D},\ t \in [0,+\infty), \\ \xi(t,\zeta) = 0, \qquad \zeta\in\partial\mathcal{D},\ t\in[0,+\infty), \\ \xi(0,\zeta) = x_0(\zeta),\ \partial_t\xi(0,\zeta) = v_0(\zeta) \qquad \zeta \in \mathcal{D}, \end{cases} \tag{25}$$

and the cost functional

$$J(0,x,u) = \mathbb{E}\int_0^{+\infty}\int_{\mathcal{D}} \left[\kappa_1(t,\zeta)(\nabla_x\xi(t,\zeta))^2 + \kappa_2(t,\zeta)\left(\frac{\partial\xi}{\partial t}(t,\zeta)\right)^2\right] d\zeta\, dt + \mathbb{E}\int_0^{+\infty}\int_{\mathcal{D}} u^2(t,\zeta)d\zeta\, dt$$

Moreover $\{\beta_i : i = 1,2...\}$ are independent standard (real valued) brownian motions defined on $(\Omega,\mathcal{F},\mathbb{P})$. We set $\mathcal{F}_t = \sigma\{\beta_i(s) : s \geq 0\ i = 1,2...\}$ and denote by $\mathcal{P}$ the predictable σ-field in $\Omega \times [0,+\infty)$.

On the coefficients we assume the following:

1. μ is a bounded measurable process defined on $([0,+\infty)\times\Omega)\times\mathcal{D}$ endowed with the σ-field $\mathcal{P}\otimes\mathcal{B}(\mathcal{D})$ with values in $\mathbb{R}^+$ (with Borel σ-field).

2. b, κ_1, κ_2 and c_i, $i=1,2,..$ are bounded measurable maps $[0,+\infty)\times\mathcal{D}\to\mathbb{R}$. We assume that κ_1 and κ_2 have values in $\mathbb{R}^+$.

3. There exists a constant $M>0$ such that:

   $\sum_{i=1}^{\infty}|c_i(t,\zeta)|^2\leq M$ for a.e. $t\in[0,+\infty)$ and a.e. $\zeta\in\mathcal{D}$.

4. there exists a constant $\alpha>0$ such that:

   $|b(t,\zeta)| \ \geq \ \alpha, \ \ |k_i(t,\zeta)| \ \geq \ \alpha$ for a.e.$(t,\zeta) \ \in \ [0,+\infty) \ \times \ \mathcal{D}, \ \ i \ = \ 1,2.$

To fit our abstract setting we let:

• $H = H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$, $U = L^2(\mathcal{D})$
• $W(t) = \sum_{i=1}^{\infty} f_i \beta_i(t)$ where $\{f_i : i = 1, 2, ...\}$ is an orthonormal basis in an arbitrary separable real Hilbert space Ξ
• $\mathcal{D}(A) = \left[H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})\right] \times H_0^1(\mathcal{D})$ and

$$\left(A\begin{pmatrix}\xi\\ v\end{pmatrix}\right)(\zeta) = \begin{pmatrix}v(\zeta)\\ \Delta_\zeta \xi(\zeta)\end{pmatrix}, \quad \begin{pmatrix}\xi\\ v\end{pmatrix} \in \mathcal{D}(A),$$

• $(B(t)u)(\zeta) = \begin{pmatrix}0\\ b(t,\zeta)u(\zeta)\end{pmatrix}, \quad \left(C_i(t)\begin{pmatrix}\xi\\ v\end{pmatrix}\right)(\zeta) = \begin{pmatrix}0\\ c_i(t,\zeta)\xi(\zeta)\end{pmatrix}$

•$\left(S(t)\begin{pmatrix}\xi\\ v\end{pmatrix}\right)(\zeta) = \begin{pmatrix}\kappa_1(t,\zeta)\xi(\zeta)\\ \kappa_2(t,\zeta)v(\zeta)\end{pmatrix}, \quad x = \begin{pmatrix}x_0\\ v_0\end{pmatrix}$

Theorem 12 *The assumptions of Theorem 9 and of Propositions 10 hold.*

Joint with F. Masiero (Milano Bicocca) Work in progress

control dependent noise state equation (finite dimension) +random perturbation:

$$\begin{cases} dy(t) = (A(t)y(t) + B(t)u(t) + f(t))\,dt + (C(t)y(t) + D(t)u(t))\,dW(t) \\ y(0) = x, \qquad t \geq 0 \end{cases} \tag{26}$$

$y \in \mathbb{R}^n$, $u \in \mathbb{R}^k$, $A, C_i \in \mathcal{M}(n \times n)$, $B, D_i \in \mathcal{M}(n \times k)$, $G \in \mathcal{M}(n \times l)$ random matrices and W and $\tilde{W}$ independent Wiener processes with values $\mathbb{R}^d$ and $\mathbb{R}^l$.

Final goal: minimize (at least in the stationary case) the following functional

$$\liminf_{\alpha \to 0} \alpha \mathbb{E} \int_0^{+\infty} e^{-2\alpha s} [\langle S(s)y(s), y(s)\rangle + |u(s)|^2]\, ds \tag{27}$$

over a suitable space of admissible controls $\mathcal{U}$.

We will assume always:

- that $S \geq \beta I$;
- that for all $t \geq 0$ and all x there exists a control u such that if $\tilde{y}^{t,x,u}$ is the solution starting at t in x of the state equation with control u and $f = 0$, then
$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} (|\sqrt{S}(s)\tilde{y}^{t,x,u}(s)|^2 + |u(s)|^2)\, ds < K \qquad \mathbb{P} - a.s. \tag{28}$$
for some positive constant K independent of t.

One has to restart from the finite horizon case since now the state equation is affine.

a) Consider the state equation (26) in $[0,T]$ with $f \in L^\infty([0,T] \times \Omega)$ and minimize

$$J_T(0,u,x) = \mathbb{E} \int_0^T [\langle S(s)y(s), y(s)\rangle + |u(s)|^2]\, ds + \mathbb{E}\langle P_T y(T), y(T)\rangle$$

b) Consider the state equation (26) in $[0,+\infty)$ with $f \in L^2([0,+\infty)\times\Omega)\cap L^\infty([0,+\infty)\times\Omega)$ and minimize

$$J_\infty(0,u,x) = \mathbb{E}\int_0^{+\infty} [\langle S(s)y(s), y(s)\rangle + |u(s)|^2]\,ds$$

c) Consider the state equation (26) in $[0,+\infty)$ with $f \in L^\infty([0,+\infty)\times\Omega)$ and minimize, for $\alpha > 0$ fixed, the functional (27). Notice that if we set $y^\alpha(s) := e^{-s\alpha}y(s),\ u^\alpha(s) := e^{-\alpha s}u(s)$, then

$$\mathbb{E}\int_0^{+\infty} e^{-2\alpha s}[\langle S(s)y(s), y(s)\rangle + |u(s)|^2]\,ds$$
$$= \mathbb{E}\int_0^{+\infty} [\langle S(s)y^\alpha(s), y^\alpha(s)\rangle + |u^\alpha(s)|^2]\,ds \quad (29)$$

Then y^α solves a state equation like (26) where A is replaced by $A-\alpha I$ and the forcing term is $f^\alpha(s) := e^{-\alpha s}f(s)$ and the results of the previous point apply.

d) Denote by J^*_α the optimal cost of the functional (29) then:

$$\liminf_{\alpha\to 0} \alpha J^*_\alpha = \inf_{u\in\mathcal{U}} \liminf_{\alpha\to 0} \alpha\mathbb{E}\int_0^{+\infty} e^{-2\alpha s}[\langle S(s)y(s), y(s)\rangle + |u(s)|^2]\,ds$$

Main difficulties

a) To cope with the affine term one has to introduce the following backward equation $t \in [0, T]$:

$$\begin{cases} dr_t^T = -H_t^* r_t^T dt - P_t f_t dt - K_t^* \cdot {g^T}_t dt + {g^T}_t \cdot dW_t, \\ r_T^T = 0. \end{cases} \tag{30}$$

The coefficients H and K are related with the coefficients of the state equation and the solution to the BSRE in $[0, T]$.

So we only know that for every $T > 0$ there exists a constant $C > 0$ such that

$$\mathbb{E}^{\mathcal{F}_\tau} \int_\tau^T |H_t|^2 dt + \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^T |K_t|^2 dt \leq C,$$

the usual fixed point technique does not work, moreover the problem is naturally multidimensional;

b) Due to the presence of the control dependent noise also the variable Q appears in the *fundamental relation*, the existence of a minimal solution for the infinite horizon Backward Riccati equation becomes more difficult to prove;

c) we have to solve also the elliptic version of the backward equation of point a), for $t \geq 0$:

$$dr_t^\infty = -[H_t^* r_t^\infty + P_t f_t + K_t^* \cdot {g^\infty}_t]dt + {g^\infty}_t \cdot dW_t,$$

In details, if we denote for $t \in [0, T]$

$$h(t, P_t, Q_t) = -[I + \sum_{i=1}^{d} \left(D_t^i\right)^* P_t D_t^i]^{-1}[P_t B_t + \sum_{i=1}^{d}(Q_t^i D_t^i + \left(C_t^i\right)^* P_t D_t^i)],^*$$

then we have: $H_t = A_t + B_t h(t, P_t, Q_t)$ and $K_t^i = C_t^i + D_t^i h(t, P_t, Q_t)$, where (P, Q) is the solution of the Riccati equation

$$\begin{aligned} dP_t = &- [A_t^* P_t + P_t A_t + \sum_{i=1}^{d}((C_t^i)^* P_t C_t^i + (C_t^i)^* Q_t + Q_t C_t^i)]dt + \sum_{i=1}^{d} Q_t^i dW_t^i + S_t\, dt \\ &+ [P_t B_t + \sum_{i=1}^{d}((C_t^i)^* P_t D_t^i + Q^i D_t^i)]h(t, P_t, Q_t)dt, \\ &P(T) = P_T, \end{aligned}$$

(Generalizations of the feedback operator considered by **A. Bensoussan and J. Frehse.[1992]**, **G.Tessitore.[1998]** when the coefficients are deterministic.)

Solutions

a) Since the backward equation (30) is linear we find a solution (r^T, g^T), $r^T \in L^\infty_{loc}([0,T] \times \Omega, \mathbb{R}^n)$ and ${g^T}_i \in L^2_{loc}([0,T] \times \Omega, \mathbb{R}^n)$, for $i = i, ..., d$. We gain regularity exploiting a duality relation with the following forward equation

$$\begin{cases} dX_s = H_s X_s ds + K_s X_s \cdot dW_s & s \in (0,T] \\ X_0 = x. \end{cases}$$

that is the closed loop equation related to the linear quadratic problem for which we can prove good estimates. This allows us to prove the necessary regularity (using estimations typical of BMO Martingale Theory) for (r^T, g^T) to get the synthesis of optimal control;

b) to build the minimal solution $(\overline{P}, \overline{Q})$ we study stability properties of the stochastic Hamilton systems that arise from the stochastic maximum principle (see also **Tang [2003]**);

c) we exploit again the duality relation between the elliptic backward equation and the closed loop equation of the infinite horizon linear quadratic problem.

We obtain a solution (r_∞, g_∞) as limit of the (r_T, g_T)'s as T tends to $+\infty$. Again the key estimates are a combination of typical bounds of BMO Martingale Theory and control interpretation.

Adapting the Theorem of Datko (that is: if the solution of a linear equation is square integrable in $(0, +\infty)$ then has exponential decay) , we prove the exponential decay of the solution of the following equation:

$$\begin{cases} dX_s = H_s X_s ds + K_s X_s \cdot dW_s \quad s > 0 \\ X_0 = x. \end{cases}$$

that is the closed loop equation of the infinite horizon case with linear state equation.

Theorem 13 *Under our assumptions + stationarity of all coefficients and taking* $f \in L^\infty(\Omega \times [0, +\infty))$ *"we have"*

$$\inf_{u \in \mathcal{U}} \liminf_{\alpha \to 0} \alpha \mathbb{E} \int_0^{+\infty} e^{-2\alpha s} [\langle S(s) y(s), y(s) \rangle + |u(s)|^2] \, ds$$

$$= \liminf_{\alpha \to 0} \alpha \int_0^{+\infty} \langle r_s^\alpha, f_s^\alpha \rangle ds +$$

$$\liminf_{\alpha \to 0} \alpha \mathbb{E} \int_0^{+\infty} |(I + \sum_{i=1}^{d} \left(D_s^i\right)^* P_s^\alpha D_s^i)^{-1} (B_s^* r_s^\alpha + D_s^* \cdot g_s^\alpha)|^2 ds.$$

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