Martingale optimality and cross hedging of insurance derivatives*

S. Ankirchner, Y. Hu, P. Imkeller, M. Müller, A. Popier, G. Reis

Université Rennes I, Humboldt-Universität zu Berlin, Université du Maine
http://wws.mathematik.hu-berlin.de/∼imkeller

La Londe Les Maures, September 11, 2007

*Supported by the DFG research center MATHEON at Berlin

1 Insurance derivatives

Weather derivatives

- underlyings: e.g. temperature, rainfall or snowfall indices
- **Aim:** transfer exogenous risk caused by fluctuations in weather patterns to capital markets

Catastrophe futures

- underlyings: e.g. loss, windspeed index (hurricane bond)
- **Aim:** transfer exogenous (insurer's) risk of abnormal losses to other parts of economy
- alternative to traditional catastrophe reinsurance

2 Hedging of portfolios of insurance derivatives

Problem: underlying is not tradable, but correlated with tradable assets

Examples:
temperature $\longleftrightarrow$ heating oil futures, electricity futures
loss index $\longleftrightarrow$ stock price of insurance companies

Mutual hedging of weather derivatives
rainfall in Spain $\longleftrightarrow$ rainfall in Scandinavia

Aims:
determine **utility indifference price**
determine explicit **derivative hedge**, i.e. optimal cross hedging strategy

describe reduction of risk by cross hedging
compare dynamic with static risk
interpret pricing by marginal utility

3 The financial market model

Index process, e.g. temperature

$$dR_t = \rho(t, R_t)dW_t + b(t, R_t)dt,$$

$b : [0, T] \times \mathbf{R}^m \to \mathbf{R}^m$, $\rho : [0, T] \times \mathbf{R}^m \to \mathbf{R}^{m \times d}$ deterministic functions, globally Lipschitz and of sublinear growth. R Markov process, $R_s^{t,r}$: start at t in r

Insurance derivative $F(R_T)$, $F : \mathbf{R}^m \to \mathbf{R}$ bounded

Correlated financial market, k risky assets with price process:

$$\frac{dS_t^i}{S_t^i} = \beta_i(t, R_t)dW_t + \alpha_i(t, R_t)dt = \beta_i(t, R_t)[dW_t + \theta_t dt], \quad i = 1, \ldots, k,$$

$\alpha : [0, T] \times \mathbf{R}^m \to \mathbf{R}^k$, $\beta : [0, T] \times \mathbf{R}^m \to \mathbf{R}^{k \times d}, \theta = \beta^*[\beta\beta^*]^{-1}\alpha$.

W $d-$dimensional Brownian motion, $\beta\beta^*$ uniformly elliptic

4 The optimal investment problem

(N. El Karoui, R. Rouge '00; J. Sekine '02; J. Cvitanic, J. Karatzas '92, Kramkov, Schachermayer '99,...)

investment strategy λ : value of portfolio fraction invested in risky assets

wealth gain on $[t, s]$

$$G_s^{\lambda,t} = \sum_{i=1}^{k} \int_t^s \lambda_u^i \frac{dS_u^i}{S_u^i} = \int_t^s \lambda_u \beta_u [dW_u + \theta_u du],$$

utility function: $U(x) = -e^{-\eta x}$ $(0 < \eta$ risk aversion); maximal expected utility from terminal wealth without and with derivative:

$$V^0(t, v, r) = \sup_{\lambda \in \mathcal{A}^{t,r}} EU(v + G_T^{\lambda,t,r}), V^F(t, v, r) = \sup_{\lambda \in \mathcal{A}^{t,r}} EU(v + G_T^{\lambda,t,r} - F(R_T^{t,r}))$$

λ^0 resp. λ^F optimal strategies

$\Delta = \lambda^F - \lambda^0$ **derivative hedge**

5 Optimization under non-convex constraints

interpretation as maximization problem with constraints

$$\pi(t,r) = \lambda(t,r)\beta(t,r) \in C(t,r) = \{x\beta(t,r) : x \in \mathbf{R}^k\}$$

here: $C(t,r)$ convex

Aim: construct solution using BSDE, even for non-convex constraints

(N. El Karoui, R. Rouge '00 for convex constraints)

$\tilde{C} \subset \mathbf{R}^k$ closed

$\tilde{\mathcal{A}}$ set of strategies λ such that

- $\lambda \in \tilde{C} \quad P \otimes l$-a.s. $\quad$ (l Lebesgue measure)

- $\{\exp(-\eta \int_0^\tau \lambda_s \frac{dS_s}{S_s}) : \tau$ stopping time in $[0,T]\}$ uniformly integrable

5 Optimization under non-convex constraints

$F = F(R_T)$ insurance derivative

For simplicity $t = 0$, $G^{\lambda,0} = G^{\lambda}$, $V(v) = V^F(0, v, r)$, etc.

First formulation:

Find

$V(v) = \sup_{\lambda \in \tilde{\mathcal{A}}} E(U(G_T^{\lambda} - F)) = \sup_{\lambda \in \tilde{\mathcal{A}}} E(U(v + \int_0^T \lambda_s \beta_s [dW_s + \theta_s ds] - F))$.

For simplicity:

$$\begin{aligned} \pi &= \lambda \beta, \\ C &= \tilde{C} \beta, \\ \mathcal{A} &= \tilde{\mathcal{A}} \beta. \end{aligned}$$

$$G_t^{\pi} = v + \int_0^t \pi_s [dW_s + \theta_s ds], \quad t \in [0, T]$$

Second formulation:

Find

$$V(v) = \sup_{\pi \in \mathcal{A}} E(U(G_T^{\pi} - F)) = \sup_{\pi \in \mathcal{A}} E(-\exp(-\eta(x + \int_0^T \pi_s [dW_s + \theta_s ds] - F))).$$

6 A solution method based on BSDE

Idea: Construct family of processes $Q^{(\pi)}$ such that

form 1
$$\begin{aligned} Q_0^{(\pi)} &= \text{constant}, \\ Q_T^{(\pi)} &= -\exp(-\eta(G_T^\pi - F)), \\ Q^{(\pi)} & \quad \text{supermartingale}, \quad \pi \in \mathcal{A}, \\ Q^{(\pi^*)} & \quad \text{martingale, for (exactly) one} \quad \pi^* \in \mathcal{A}. \end{aligned}$$

Then

$$\begin{aligned} E(-\exp(-\eta[G_T^\pi - F])) &= E(Q_T^{(\pi)}) \\ &\le E(Q_0^\pi) \\ &= V(v) \\ &= E(Q_0^{(\pi^*)}) \\ &= E(-\exp(-\eta[G_T^{(\pi^*)} - F])). \end{aligned}$$

Hence π^* optimal strategy.

6 A solution method based on BSDE

Introduction of BSDE into problem

Find generator f of BSDE

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad Y_T = F,$$

such that with

$$Q_t^{(\pi)} = -\exp(-\eta[G_t^\pi - Y_t]), \quad t \in [0, T],$$

we have

$$\begin{aligned} Q_0^{(\pi)} &= -\exp(-\eta(v - Y_0)) \\ &= \text{constant}, \end{aligned}$$
(fulfilled)

form 2 $$Q_T^{(\pi)} = -\exp(-\eta(G_T^\pi - F))$$ (fulfilled)

$Q^{(\pi)}$ supermartingale, $\pi \in \mathcal{A}$,
$Q^{(\pi^*)}$ martingale, for (exactly) one $\pi^* \in \mathcal{A}$.

This gives solution of valuation problem.

7 Construction of generator of BSDE

How to determine f:

Suppose f generator of BSDE. Then

$$\begin{aligned} Q_t^{(\pi)} &= -\exp(-\eta[G_t^\pi - Y_t]) \\ &= -\exp(-\eta[v - Y_0]) \cdot \exp(-\eta[\int_0^t (\pi_s - Z_s)dW_s - \int_0^t [f(s, Z_s) - \pi_s\theta_s]ds]) \\ &= \exp(-\eta[v - Y_0]) \cdot \exp(-\eta \int_0^t (\pi_s - Z_s)dW_s - \frac{\eta^2}{2}\int_0^t (\pi_s - Z_s)^2 ds) \\ &\quad \cdot - \exp(\int_0^t [\eta f(s, Z_s) - \eta\pi_s\theta_s + \frac{\eta^2}{2}(\pi_s - Z_s)^2]ds) \\ &= M_t^{(\pi)} \cdot A_t^{(\pi)}, \end{aligned}$$

with $M^{(\pi)}$ nonnegative martingale. $Q^{(\pi)}$ satisfies **(form 2)** iff for

$$q(\cdot, \pi, z) = f(\cdot, z) - \pi\theta + \frac{\eta}{2}(\pi - z)^2, \quad \pi \in \mathcal{A}, z \in \mathbb{R},$$

we have

7 Construction of generator of BSDE

form 3

$$
\begin{aligned}
q(\cdot,\pi,z) &\geq 0, \quad \pi\in\mathcal{A} \quad \text{(supermartingale cond.)}\\
q(\cdot,\pi^*,z) &= 0, \quad \text{for (exactly) one} \quad \pi^*\in\mathcal{A} \quad \text{(martingale cond.)}.
\end{aligned}
$$

Now

$$
\begin{aligned}
q(\cdot,\pi,z) &= f(\cdot,z)-\pi\theta+\frac{\eta}{2}(\pi-z)^2\\
&= f(\cdot,z)+\frac{\eta}{2}(\pi-z)^2-(\pi-z)\cdot\theta+\frac{1}{2\eta}\theta^2 \quad -z\theta-\frac{1}{2\eta}\theta^2\\
&= f(\cdot,z)+\frac{\eta}{2}[\pi-(z+\frac{1}{\eta}\theta)]^2 \quad -z\theta-\frac{1}{2\eta}\theta^2.
\end{aligned}
$$

Under **non-convex constraint** $\pi\in C$:

$$
[\pi-(z+\frac{1}{\eta}\theta)]^2\geq d^2(C,z+\frac{1}{\eta}\theta).
$$

with **equality** for at least one possible choice of p^* due to **closedness** of C.
Hence **(form 3)** is solved by the choice

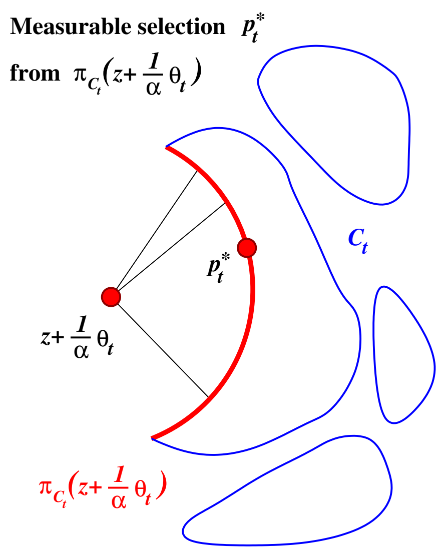
7 Construction of generator of BSDE

form 4

$$f(\cdot, z) = -\frac{\eta}{2} d^2(C, z + \frac{1}{\eta}\theta) + z \cdot \theta + \frac{1}{2\eta}\theta^2 \quad \text{(supermartingale)}$$

π^* such that $d(C, z + \frac{1}{\eta}\theta) = d(\pi^*, z + \frac{1}{\eta}\theta)$ (martingale).

Problem: Let
$\Pi_C(v) = \{\pi \in \mathbb{R}^d : d(C, v) = d(\pi, v)\}$. Find measurable selection π_t^* from $\Pi_{C_t}(Z_t + \frac{1}{\eta}\theta_t)$. Solved by classical **measurable selection method**.



8 Main result

Thm 1
(Y, Z) unique solution of BSDE

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad t \in [0, T],$$

with

$$f(t, Z_t) = -\frac{\eta}{2} d^2(C_t, Z_t + \frac{1}{\eta}\theta_t) + Z_t \cdot \theta_t + \frac{1}{2\eta}\theta_t^2.$$

Then value function of utility optimization problem under constraint $\pi \in \mathcal{A}$ given by

$$V(v) = -\exp(-\eta[v - Y_0]).$$

There exists an (non-unique) optimal trading strategy $\pi^* \in \mathcal{A}$ such that

$$\pi_t^* \in \Pi_{C_t}(Z_t + \frac{1}{\eta}\theta_t), \quad t \in [0, T].$$

Proof:

- existence, uniqueness for BSDE with quadratic non-linearity in z (M. Kobylanski '00)
- measurable selection theorem for $\Pi_{C_t}(Z_t + \frac{1}{\eta}\theta_t)$
- BMO properties of the martingales $\int Z_s dW_s, \int \pi_s^* dW_s$ for uniform integrability of exponentials (regularity of coefficients)•

9 Calculation of derivative hedge

generalization to $[t, T]$ instead of $[0, T]$, cond. on $R_t = r$:

$(Y^{t,r}, Z^{t,r})$,$\pi^{t,r}$ (without F) resp. $(\hat{Y}^{t,r}, \hat{Z}^{t,r})$,$\hat{\pi}^{t,r}$ (with F) instead of (Y, Z),π yields

$$V^0(t, v, r) = -\exp(-\eta(v - Y_t^{t,r})), \quad V^F(t, v, r) = -\exp(-\eta(v - \hat{Y}_t^{t,r})),$$

instead of $V(v) = -\exp(v - Y_0)$.

due to linearity of $C(t, r)$ projections unique and linear, hence

$$\pi_s^{t,r} = \Pi_{C(t,r)}[Z_s^{t,r} + \frac{1}{\eta}\theta(s, R_s^{t,r})], \quad \hat{\pi}_s^{t,r} = \Pi_{C(t,r)}[\hat{Z}_s^{t,r} + \frac{1}{\eta}\theta(s, R_s^{t,r})],$$

and so

$$\Delta\beta(s, R_s^{t,r}) = \Pi_{C(t,r)}[\hat{Z}_s^{t,r} - Z_s^{t,r}].$$

10 Markov property and its consequences

Markov property of R implies (Kobylanski '00, El Karoui, Peng, Quenez '97):

Thm 2
There are measurable (deterministic) functions u and $\hat{u}$ such that

$$Y_s^{t,r} = u(s, R_s^{t,r}), \quad \hat{Y}_s^{t,r} = \hat{u}(s, R_s^{t,r}).$$

There are measurable (deterministic) functions v and $\hat{v}$ such that

$$Z_s^{t,r} = v\rho(s, R_s^{t,r}), \quad \hat{Z}^{t,r} = \hat{v}\rho(s, R_s^{t,r}).$$

Corollary 1

$$p(t,r) := Y_t^{t,r} - \widehat{Y}_t^{t,r} = u(t,r) - \hat{u}(t,r)$$

is the **indifference price**, i.e. $V^F(t, v - p(t,r), r) = V^0(t, v, r)$.

p depends only on R, not on S

Aim: Explicit description of Δ

11 Differentiability

Thm 3 (**Parameter Differentiability**) smoothness conditions on F, f
There exists a version of $(\widehat{Y}_s^{t,r}, \widehat{Z}_s^{t,r})$ such that a.s.
- $\widehat{Y}_s^{t,r}$ is continuous in s and cont. differentiable in r (classical sense)
- $\widehat{Z}^{t,r}$ is differentiable in a weak sense (norm topology)
- $(\nabla_r \widehat{Y}^{t,r}, \nabla_r \widehat{Z}^{t,r})$ solves the BSDE

$$\begin{aligned}\nabla_r \widehat{Y}_t^r &= \nabla_r F(R_s^{t,r}) \nabla_r R_s^{t,r} - \int_t^T \nabla_r \widehat{Z}_s^{t,r} dW_s \\ &\quad + \int_t^T \Big[\nabla_r f(s, R_s^{t,r}, \widehat{Z}_s^{t,r}) \nabla_r R_s^{t,r} \\ &\qquad + \nabla_z f(s, R_s^{t,r}, \widehat{Z}_s^{t,r}) \nabla_r \widehat{Z}_s^{t,r} \Big] ds.\end{aligned}$$

Proof uses norm inequalities, and inverse Hölder inequalities, based on BMO properties of the stochastic integral processes of $\hat{Z}^{t,r}$

Thm 4 (Malliavin Differentiability)

$$D_\vartheta \widehat{Y}_s^{t,r} = \nabla_r \widehat{u}(s, R_s^{t,r}) D_\vartheta R_s^{t,r}$$

and

$$\widehat{Z}_s^{t,r} = D_s \widehat{Y}_s^{t,r} = \nabla_r \widehat{u}(s, R_s^{t,r}) \rho(s, R_s^{t,r})$$

12 Explicit description of derivative hedge

Properties of the BSDEs $\Longrightarrow$

Thm 5
The **indifference price** $p(t,r) = Y_t^{t,r} - \widehat{Y}_t^{t,r}$ is differentiable in r.

Thm 6
The **derivative hedge** Δ at time t depends only on R_t, and

$$\begin{aligned} \Delta(t,r)\beta(t,r) &= \Pi_{C(t,r)}[\widehat{Z}_t^{t,r} - Z_t^{t,r}] \\ &= \Pi_{C(t,r)}[\nabla_r(\widehat{Y}_t^{t,r} - Y_t^{t,r})\rho(t,r)] \\ &= -\Pi_{C(t,r)}[\nabla_r p(t,r)\rho(t,r)]. \end{aligned}$$

Remarks:

- complete case: $\Delta =$ 'delta hedge'
- where is the risk aversion η?

13 Example: Heating degree days

- common underlying of weather derivatives
- $T_i =$ average of the maximum and the minimum temperature on day i at a specific location
- $HDD_i = \max(0, 18 - T_i)$

Cumulative heating degree days

$$cHDD_t = \sum_{i=1}^{30} HDD_{t-i}$$

Derivatives:

- Option: $(cHDD - K)^+$
- Swap: $b(cHDD - K)$

13 Example: Heating degree days

$cHDD$:

- statistical analysis shows: cHDDs are log-normally distributed (M. Davis '01)

- cHDD can be modeled as a *geometric Brownian motion*

$$dX_t = \mu X_t dt + \nu X_t dW_t$$

(moving average)

Other indices: cooling degree days

$$CDD_i = \min\left(0, 18 - T_i\right)$$

13 Example: Heating degree days

- $R =$ cHDDs (geometric Brownian Motion)
- $d = 2$
- 1-dim market + index: $k = m = 1$
- index volatility: $\rho = \begin{pmatrix} \alpha & 0 \end{pmatrix}$
- price volatility: $\beta = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$ with $\alpha, \beta_1, \beta_2 \in \mathbf{R} \setminus \{0\}$

Then

$$\Delta(t, r) = -\alpha \frac{\partial p(t, r)}{\partial r} \frac{\beta_1}{\beta_1^2 + \beta_2^2}.$$

14 Example: Heating degree days; diversification pressure

derivative hedge:

$$\Delta(t,r) = -\alpha \frac{\partial p(t,r)}{\partial r} \frac{\beta_1}{\beta_1^2 + \beta_2^2}.$$

Call option: $F(R_T) = (R_T - K)^+$

$\Longrightarrow \frac{\partial p(t,r)}{\partial r} > 0$

Comparison of the optimal strategies:

- $\beta_1 \alpha < 0$ (negative correlation)
  $\Longrightarrow \quad F(X_T)$ diversifies portfolio $\Longrightarrow \quad \Delta > 0$
  $\Longrightarrow \quad \widehat{\pi} > \pi$
- $\beta_1 \alpha > 0$ (positive correlation)
  $\Longrightarrow \quad F(X_T)$ amplifies portfolio $\Longrightarrow \quad \Delta < 0$
  $\Longrightarrow \quad \widehat{\pi} < \pi$