Exponential growth of random Fibonacci sequences

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DE LA RECHERCHE
SCIENTIFIQUE

UNIVERSITÉ DE ROUEN

Viswanath's result

$$F_1 = F_2 = 1, \qquad F_{n+1} = F_n \pm F_{n-1} \quad \forall n \geq 2$$

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$$\log(1,13198824\ldots) = \frac{1}{4}\int_{-\infty}^{+\infty} \log \frac{1+4m^4}{(1+m^2)^2} d\nu(m)$$

where ν is a probability measure inductively defined on Stern-Brocot intervals.

Generalizations

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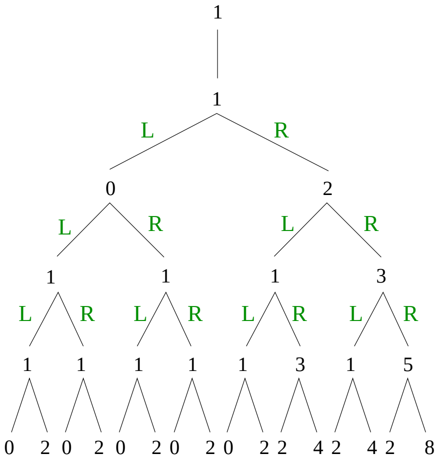
Non-linear case

$$(F_{n-1}, F_n) = (F_1, F_2) X_3 X_4 \dots X_n$$

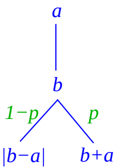
where $X_i = R$ with probability p and $X_i \in \{L_1, L_2\}$ with probability $1 - p$

$$R := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad L_1 := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad L_2 := \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Reduction

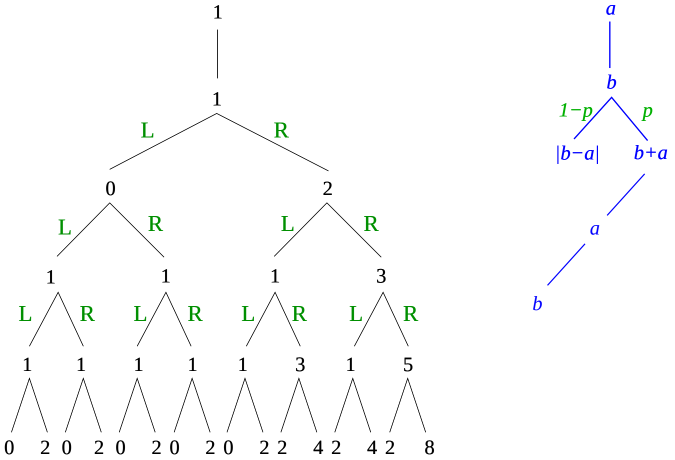


1
1
L R
0 2
L R L R
1 1 1 3
L R L R L R L R
1 1 1 1 1 3 1 5
0 2 0 2 0 2 0 2 0 2 2 4 2 4 2 8

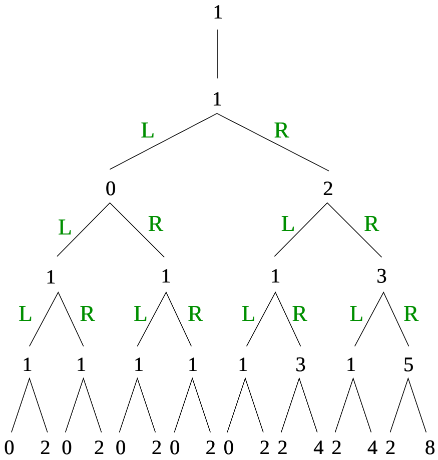


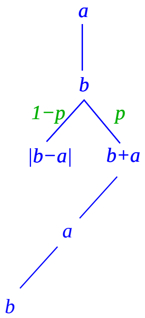
a
b
1−p
p
|b−a|
b+a

Reduction


1
1
L
R
0
2
L
R
L
R
1
1
1
3
L
R
L
R
L
R
L
R
1
1
1
1
1
3
1
5
0
2
0
2
0
2
0
2
0
2
2
4
2
4
2
8
a
b
1−p
p
|b−a|
b+a
a
b

Reduction

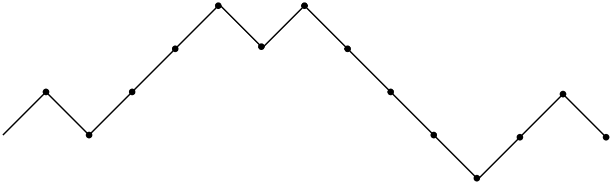




$$RL_1L_2 = \mathsf{Id}$$

Reduction

We can remove all patterns RLL in the infinite sequence coding the path.



$$RLRRRLRLLLLRRL\dots$$

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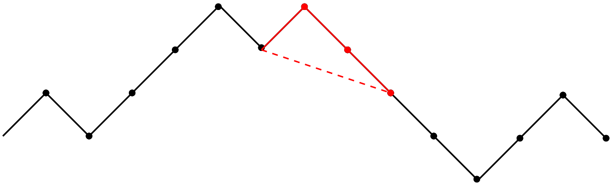

$$RLRRRL\color{red}{RLL}LLRRL\ldots$$

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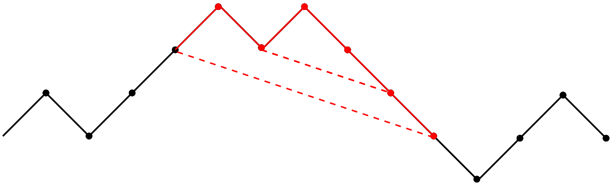

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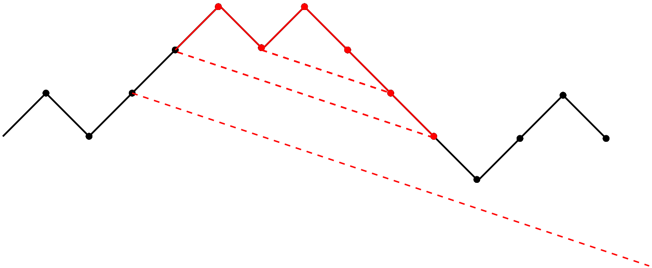

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If $p > 1/3$, infinitely many R's survive.

Law of the reduced sequence ($p > 1/3$)

$L^s B_1 B_2 B_3 \ldots$ with $B_i \in \{R, RL\}$

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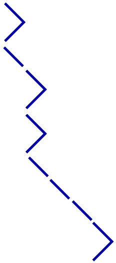

$$P(B_1 = R) = \alpha := \frac{2p}{p + \sqrt{p(4-3p)}} \in [1/2, 1]$$

Quotients

Let $(G_\ell) = (F_{n_\ell})$ be the labels read in the tree along the reduced sequence.

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How do the quotients evolve when we add a new block?

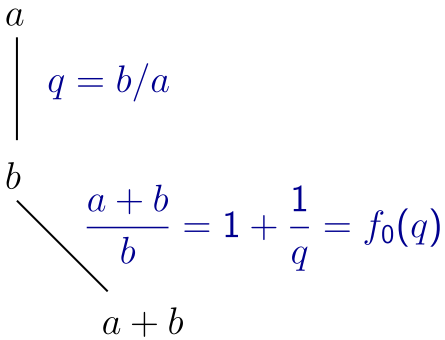
Quotients

a

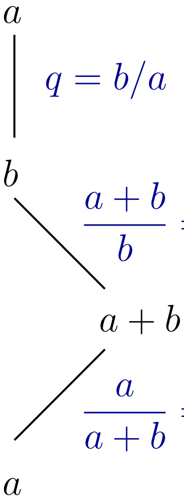
$q = b/a$

b

Quotients



Quotients



$$q = b/a$$

$$\frac{a+b}{b} = 1 + \frac{1}{q} = f_0(q)$$

$$\frac{a}{a+b} = \frac{1}{1+q} = f_1(q)$$

The measure ν_α

$$f_1(q) = \frac{1}{1+q} \qquad f_0(q) = 1 + \frac{1}{q}$$

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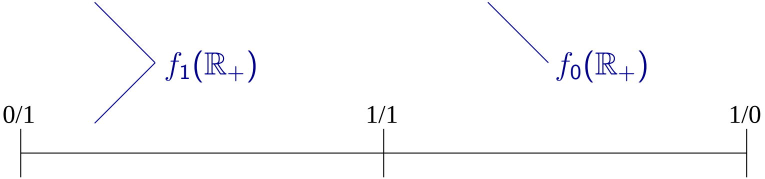

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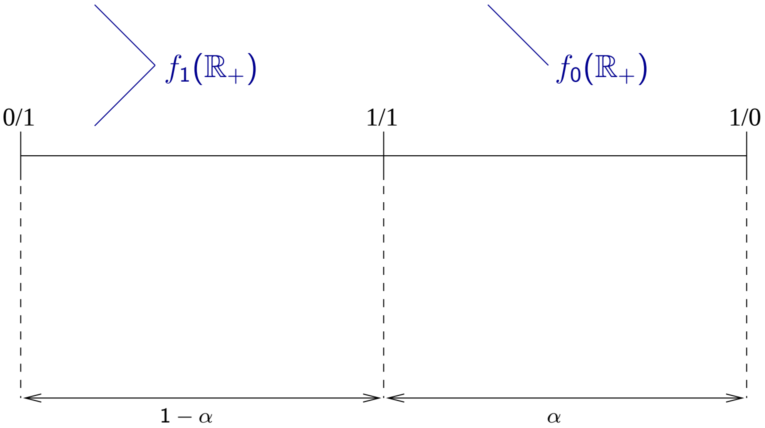

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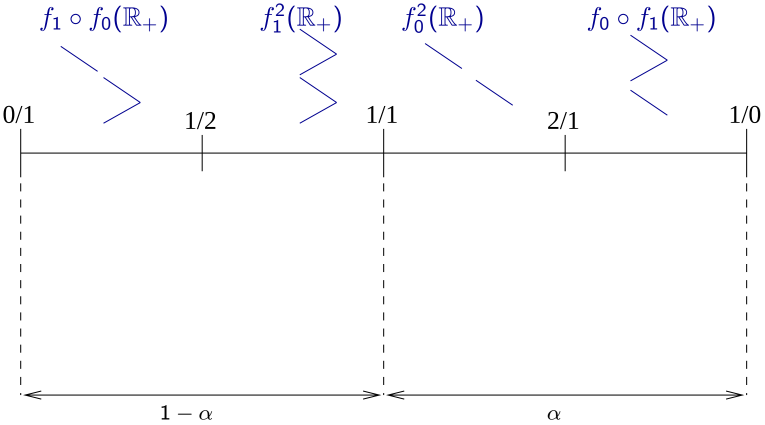

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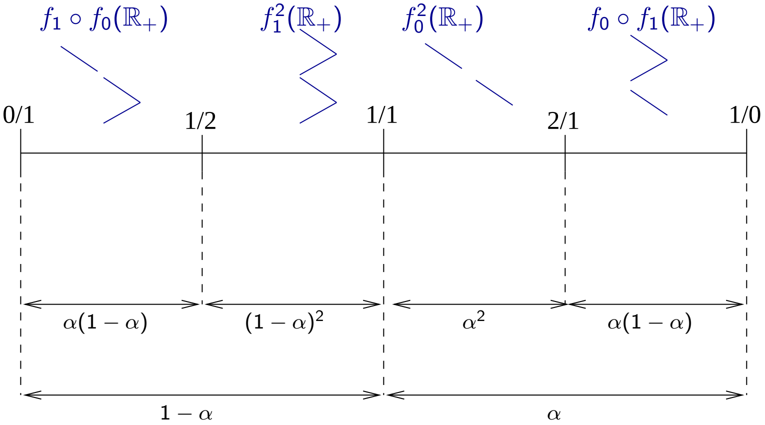

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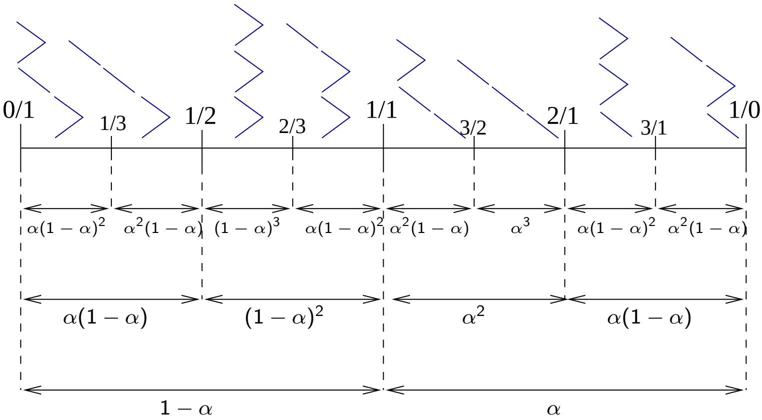

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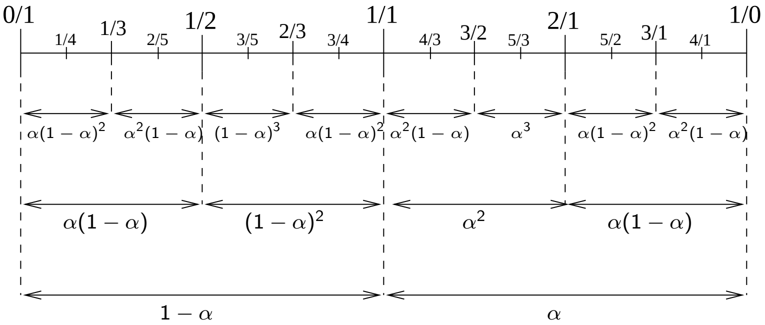


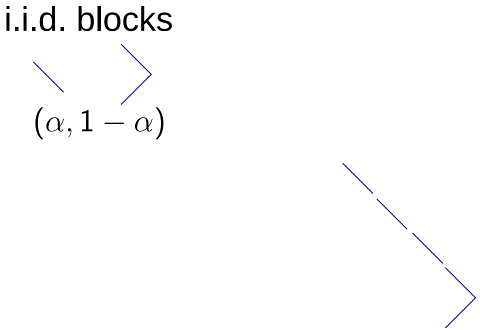
$$\left[\frac{a}{b}, \frac{c}{d}\right] \longrightarrow \left[\frac{a}{b}, \frac{a+c}{b+d}\right], \left[\frac{a+c}{b+d}, \frac{c}{d}\right]$$

Stern-Brocot intervals

Every rational number appears as the endpoint of a SB interval:

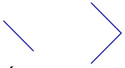
$$\left[\frac{a}{b}, \frac{c}{d}\right] \text{ with } ad - bc = -1$$

The measure ν_α

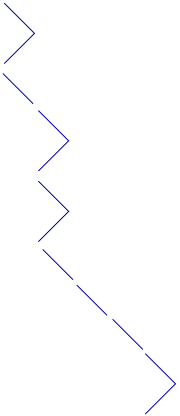


The measure ν_α

i.i.d. blocks



$(\alpha, 1-\alpha)$



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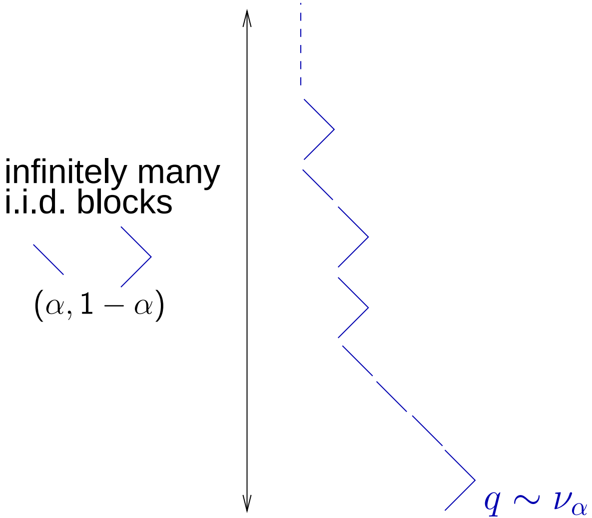

Law of the quotient F_{n+1}/F_n

$X_1 X_2 \dots X_n$
$\longrightarrow \text{reduction}(X_1 X_2 \dots X_n) = L^s B_1 B_2 \dots B_i$

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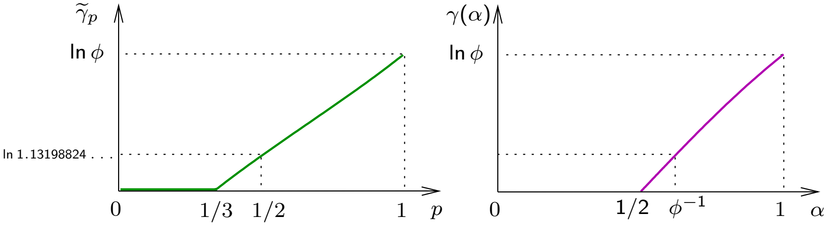
$$\frac{1}{n} \log F_n = \frac{1}{n} \sum_{i=1}^{n-1} \log \frac{F_{i+1}}{F_i} \xrightarrow[n \to +\infty]{} \int_0^{+\infty} \log q \, d\nu_\alpha(q) \text{ a.s.}$$

Lyapunov exponent



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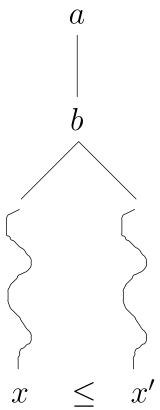

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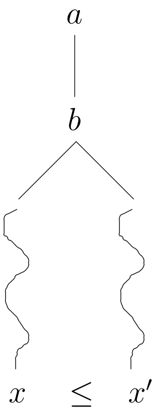
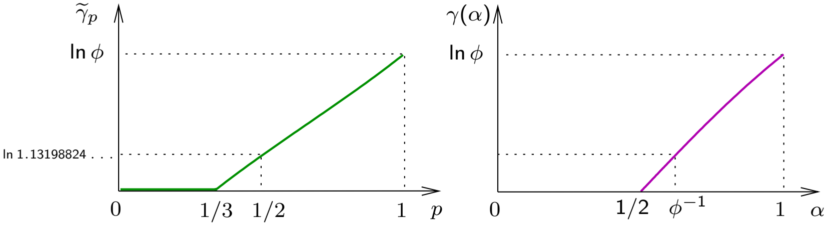
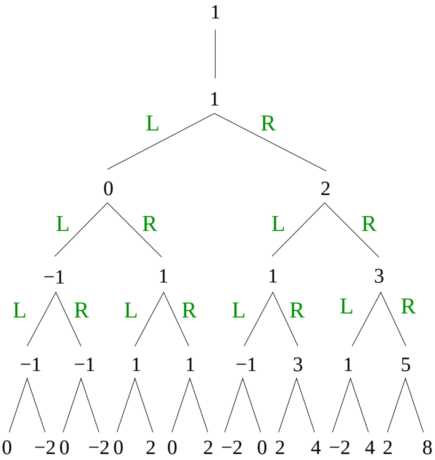


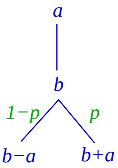
Coupling of two paths with parameters $p \leq p'$

Lyapunov exponent



Linear case





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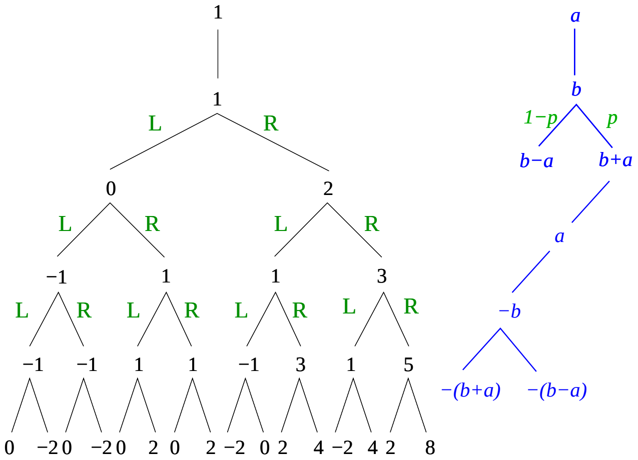

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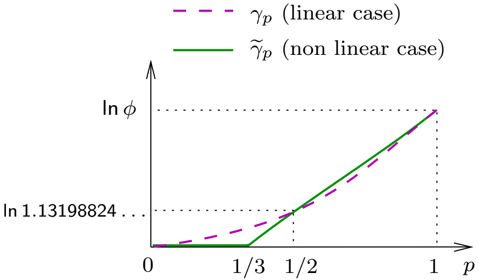
Linear case

$$\forall p, \quad \frac{1}{n} \log F_n \xrightarrow[n \to +\infty]{} \int_0^{+\infty} \log q \; d\nu_\alpha(q) \text{ a.s.}$$

$$\alpha = \frac{3p - 2 + \sqrt{5p^2 - 8p + 4}}{2p} \in [1/2, 1]$$

Lyapunov exponents



Generalizations

$$\begin{aligned} F_{n+1} &= \lambda F_n \pm F_{n-1} \quad \text{(linear case)} \\ F_{n+1} &= |\lambda F_n \pm F_{n-1}| \quad \text{(non-linear case)} \end{aligned}$$

where the $+$ sign is chosen with probability p, and $\lambda = \lambda_k = 2\cos(\pi/k)$, $k \geq 3$.

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$\lambda_3 = 1$

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$$\lambda_3 = 1, \lambda_4 = \sqrt{2}, \lambda_5 = \frac{1+\sqrt{5}}{2}, \lambda_6 = \sqrt{3}, \ldots$$

Generalizations

$$R := \begin{pmatrix} 0 & 1 \\ 1 & \lambda_k \end{pmatrix} \quad L_1 := \begin{pmatrix} 0 & -1 \\ 1 & \lambda_k \end{pmatrix} \quad L_2 := \begin{pmatrix} 0 & 1 \\ 1 & -\lambda_k \end{pmatrix}$$

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$$a_0\lambda_k + \cfrac{1}{a_1\lambda_k + \cfrac{1}{\ddots + \cfrac{1}{a_n\lambda_k + \ddots}}}$$