Backward doubly stochastic differential equations with quadratic growth and applications to quasilinear SPDEs

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La Londe 14 september 2007

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- $$\begin{aligned} -\frac{\partial u}{\partial t} + Lu - F(t, x, u, \sigma(t, x) Du) &= 0 \text{ in } (0, T) \times \mathbb{R}^n \\ u(T, x) &= g(x) \quad \text{in } \mathbb{R}^n. \end{aligned}$$

  $$Lu = -\frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i,j=1}^{n} b_i(t, x) \frac{\partial u}{\partial x_i},$$

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- $$\begin{aligned} u(t,x) &= h(x) + \int_s^T \left[\mathcal{L}_s u(s,x) + f(x, u(s,x), (\nabla u \sigma)(s,x))\right] ds \\ &\quad + \int_s^T g(x, u(s,x), (\nabla u \sigma)(s,x))\, dB_s \end{aligned}$$

  where $u \in R^k$

  $$(\mathcal{L}u)(t,x) = (Lu_i)(t,x) \qquad 1 \leq i \leq k$$

  and

  $$L = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$$

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Introduction

Backward doubly stochastic differential equation (BDSDE) are equation of the following type :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s)\, dB_s - \int_t^T Z_s dW_s, \quad 0 \le t \le T \tag{1.1}$$

where the dW is a forward Itô integral and the dB is a backward Itô integral.

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- In the present note, we consider the case where g is uniformly lipischitz and f is continuous and quadratic growth in z. We prove the existence a minimal and a maximal solution.

Let

$$f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\rightarrow\mathbb{R}$$
$$g:\Omega\times[0,T]\times\mathbb{R}\rightarrow\mathbb{R}$$

be measurable and such that for any $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,

$$f(.,y,z)\in M^2\left(0,T,\mathbb{R}^d\right)$$
$$g(.,y)\in M^2(0,T,\mathbb{R})$$

Moreover, we assume that there exist constants $C>0$ and $K>0$, such that for any $(\omega,t)\in\Omega\times[0,T]$ and $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,

$$\left\{\begin{array}{c} |f(t,y,z)|\leq C\left(1+|z|^2\right) \\ |g(t,y_1)-g(t,y_2)|\leq C|y_1-y_2| \\ g(t,y)\leq K \end{array}\right. \tag{H1}$$

let us assume that $\xi\in S_T^\infty([0,T],\mathbb{R})$.

BDSDE with quadratic growth

Definition

A pair of process $(y, z) : \Omega \times [0, T] \longrightarrow \mathbb{R} \times \mathbb{R}^d$ is solution of BDSDE (1.1) if $(y, z) \in S_T^{\infty}([0, T]; \mathbb{R}) \times M_T^2(0, T; \mathbb{R}^d)$, and satisfies BDSDE (1.1).

Existence result

Theorem

Under Assumption $(H1)$, *BDSDE* (1.1) *has a solution* $(Y, Z) \in S_T^{\infty}([0, T], \mathbb{R}) \times M_T^2(0, T, \mathbb{R}^d)$. *Moreover, this equation admits a minimal solution* (Y_*, Z_*) *and a maximal solution* (Y^*, Z^*).

Uniqueness result

We say that the coefficient f satisfies condition (H1)-(H2) with some constants $C,\ C_1 > 0$, if for every $t \in \mathbb{R}^+$,

$$\begin{cases} |f(t,y,z)| \leq C\left(1+|z|^2\right), & P\ p.s \\ \left|\frac{\partial f}{\partial z}(t,y,z)\right| \leq C(1+|z|) & \end{cases} \tag{H2}$$

$$\left|\frac{\partial f}{\partial y}(t,y,z)\right| \leq C_1\left(1+|z|^2\right), \quad P\ p.s. \tag{H3}$$

Theorem

Under Assumptions (H2)-(H3), BDSDE (1.1) *has a uinque solution in* $S_T^{\infty}([0,T],\mathbb{R}) \times M_T^2(0,T,\mathbb{R}^d)$.

Let $b \in C^3\left(\mathbb{R}^d, \mathbb{R}^d\right)$, and $\sigma \in C^3\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right)$. For $(t,x) \in [0,T] \times \mathbb{R}^d$, let $\{X_s^{t,x}, t \leq s \leq T\}$ denote the unique solution of the following SDE :

$$\begin{aligned} dX_s^{t,x} &= b\left(X_s^{t,x}\right) ds + \sigma\left(X_s^{t,x}\right) dW_s, \qquad t \leq s \leq T \qquad (1.2)\\ X_s^{t,x} &= x, \end{aligned}$$

Let $h \in C^2\left(\mathbb{R}^d, \mathbb{R}\right)$. For $(t,x) \in [0,T] \times \mathbb{R}^d$, let $\left\{\left(Y_s^{t,x}, Z_s^{t,x}\right); t \leq s \leq T\right\}$ denote the unique solution of the following BDSDE:

$$\begin{aligned} Y_s^{t,x} &= h\left(X_T^{t,x}\right) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr + \int_s^T g\left(r, X_r^{t,x}, Y_r^{t,x}\right) dB_r \\ &\quad - \int_s^T Z_r^{t,x} dW_s \qquad (1.3) \end{aligned}$$

BDSDE and quasilinear SPDEs

We will relate BDSDE (1.3) to the following quasilinear backward stochastic partial differential equations:

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T \left[\mathcal{L}u(s,x) + f(s,x,u(s,x),(\nabla u\sigma)(s,x))\right] ds \\ &\quad + \int_t^T g(s,x,u(s,x))\, dB_s \end{aligned} \tag{1.4}$$

where $u : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$, with
$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i}$.

Theorem

Let f, g satisfy assumption (H1). Let $\{u(t,x),(t,x) \in [0,T] \times \mathbb{R}^d\}$ be a random field such that $u(t,x)$ is $\mathcal{F}_{t,T}^B-$measurable for each (t,x), $u \in C^{0,2}\left([0,T] \times \mathbb{R}^d, \mathbb{R}\right)$ a.s, and u satisfies equation (1.4). Then $u(t,x) = Y_t^{t,x}$, where $\{(Y_s^{t,x}, Z_s^{t,x}); t \le s \le T\}_{t \ge 0, x \in \mathbb{R}^d}$ is the unique solution of the BDSDE (1.3).

Proposition

Let $(Y, Z) \in \mathcal{S}_T^{\infty}(\mathbb{R}) \times M_T^2(\mathbb{R}^d)$ *be a solution of BDSDE (1.1), with parameters* (f, g, ξ)*, and suppose that* f *satisfies (H1). Then for all* $0 \leq t \leq T$,

$$Y_t \leq E\left[\left(\sup_{\Omega} Y_T\right)^{+} / \mathcal{F}_t\right] + C(T-t) \qquad a.s$$

$$(\ resp\ .\ \ Y_t \geq E\left[\left(\inf_{\Omega} Y_T\right)^{-} / \mathcal{F}_t\right] - C(T-t) \qquad a.s).$$

Moreover, there exists a constant $\widetilde{C}$ *depending on* $\|Y\|_{\infty}$ *and* C *such that*

$$E\int_0^T |Z_s|^2\, ds \leq \widetilde{C}.$$

Proof of the existence

Consider the solution φ_t of the stochastic differential equation :

$$\varphi_t = \left(\sup_{\Omega} Y_T\right)^+ + \int_t^T C ds + \int_t^T g\left(s, \varphi_s\right) dB_s \qquad C > 0.$$

Our aim is to prove that $Y_t \leq \varphi_t$. Applying Itô's formula to the process $Y_t - \varphi_t$, with the increasing C^2 function G given by

$$G\left(u\right) = \begin{cases} e^{2Cu} - 1 - 2Cu - 2C^2u^2 & \text{for } u \in [0, M] \\ 0 & \text{for } \quad u \in [-M, 0] \end{cases}$$

where $M = \|Y\|_\infty + \|\varphi\|_\infty$, Applying Gronwall's lemma we get:

$$EG\left(Y_t - \varphi_t\right) = 0, \qquad \forall t \in [0, T],$$

We obtain $Y_t \leq \varphi_t \quad P-\ a.s.$ Hence,

$$Y_t \leq E\left[\left(\sup_{\Omega} Y_T\right)^+ / \mathcal{F}_t\right] + C\left(T - t\right) = M^+.$$

Proof of the existence

The second inequality will be proved by using similar arguments. Indeed, let ψ be the solution of the SDE :

$$\psi_t = \left(\inf_{\Omega} Y_T\right)^{-} - \int_t^T C ds + \int_t^T g\left(s, \psi_s\right) dB_s.$$

This implies that $Y_t \geq \psi_t, \quad P-\ a.s$, and then

$$Y_t \geq E\left[\left(\inf_{\Omega} Y_T\right)^{-} / \mathcal{F}_t\right] - C\left(T-t\right) = M^{-}. \qquad P-\ a.s.$$

We shall prove the inequality $E\int_0^T |Z_s|^2 ds \leq \widetilde{C}$. Let $M = \|Y\|_{\infty}$ and the function H defined on $[-M, M]$ by

$$H\left(y\right) = \frac{1}{2C^2}\left[e^{2C\left(y+M\right)} - \left(1 + 2C\left(y+M\right)\right)\right],$$

Proof of the existence

Ito's formula applied to $H\left(Y_t\right)$ it holds that:

$$EH\left(Y_0\right)+E\int_0^T\left|Z_s\right|^2ds\leq H\left(M\right)+M^2Ce^{4CM}T,$$

which leads $E\int_0^T\left|Z_s\right|^2ds\leq H\left(M\right)+M^2Ce^{4CM}T=\widetilde{C}$

Proof of the existence

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Proof of the existence

Let us consider the following BDSDE

$$X_t = \eta + \int_t^T F(s, X_S, \Lambda_s)\, ds + \int_t^T G(s, X_s)\, dB_s - \int_t^T \Lambda_s dW_s. \quad (1.5)$$

Theorem

Assume the following hypotheses.
(i) η be bounded and $\mathcal{F}_T-$measurable random variable with values in $\mathbb{R}$.
(ii) $F : \Omega \times [0, T] \times]0, \infty[\times \mathbb{R}^d \rightarrow \mathbb{R}$ a $\mathcal{P}-$measurable function, continuous in (x, λ) and satisfying the following structure condition :

$$\exists C > 0 \text{ tel que } \forall t, x, \lambda \quad P \ p.s \quad -2C^2x - C|\lambda|^2 \leq F(s, x, \lambda) \leq 2C^2x$$

and

$$|G(t, x)| \leq 2CK|x|.$$

Then BDSDE (1.5) has a maximal solution (X_t, Λ_t).

Following Proposition 3, we have $M^- \leq Y_t \leq M^+$, then
$m = e^{2cM^-} \leq X_t \leq e^{2cM^+} = M$
Step 1 Existence of X_t.
Lett $\gamma_p : \mathbb{R}^d \to \mathbb{R}$ be a smooth function satisfying :

$$\gamma_p(\lambda) = \begin{cases} 1 & \text{si } |\lambda| \leq 1 \\ 0 & \text{si } |\lambda| \geq 1 \end{cases}$$

let us define the function

$$F_p(t,x,\lambda) = 2c^2x\left(1-\gamma_p(\lambda)\right) + \gamma_p(\lambda)\,F(t,x,\lambda).$$

we have $\lim_{p\to\infty} \searrow F_p = F$.

Proof of the existence

Applying theorem the existence in Y. Shi et al. the BDSDE associated with (F_p, G, η) has a solution (X^p, Λ^p). Again by applying the comparison theorem proved in Y. Shi et al., it follows that $M \geq X^p \geq X^{p+1} \geq m$. Since $F_{p+1} \leq F_p$, there exists a process X_t such that

$$X_t = \lim_{p \to \infty} X_t^p \quad \forall t \leq T \ , \qquad P \ p.s.$$

In addition $P - a.s$, for all $t \leq T, M \geq X_t \geq m$.

Proof of the existence

Step 2. There exists a subsequence of $(\Lambda^p)_{p\geq 1}$ which converges in M^2. The sequence $(\Lambda^p)_{p\geq 1}$ is bounded in M^2 by proposition 3, then there exists a subsequence of $(\Lambda^p)_{p\geq 1}$, which we still denote by $(\Lambda^p)_{p\geq 1}$, which converges weakly in M^2 to a process $\Lambda := (\Lambda_t)_{t\leq T}$.
Let $\theta = \max\left(\frac{1}{m}, 4c^2M\right)$, $\phi(x) = \left(\frac{e^{12\theta x}-1}{12\theta}\right) - x$ and $p \leq q$, then we have $X^p \geq X^q$. On the other hand, by using Itô's formula with $\phi(X^p - X^q)$ the fact $(\Lambda^p)_{p\geq 1}$ converge weakly in M^2, from which we deduce that

$$\lim_{p\to\infty} E\int_t^T |\Lambda_s^p - \Lambda_s|^2 \, ds = 0.$$

Hence $(\Lambda_s^p)_{p\geq 1}$ converges to Λ_s.

Step 3. The process $X=(X_t)_{t\leq T}$ is continuous.
We shall prove that (X^p) converges uniformly to X in L^2. Let $p\leq q$, applying Itô's formula to $(X^p-X^q)^2$, and applying Burkholder-Davis-Gundy inequality, aqnd the fact F and F_p are continuous, and (F_p) is a decreasing sequence converging to F, then by Dini's theorem $(F_p(t,.,.))$ converges to $F(t,.,.)$ uniformly. On the other hand, since (Λ^p) converge in M^2 to Λ, then there exists $\widetilde{\Lambda}\in M^2$ and a subsequence, which we still denote $(\Lambda^p)_{p\geq 1}$ such that (Λ^p) converge to Λ and $\sup_{p\geq 1}|\Lambda_t^p|\leq\widetilde{\Lambda}_t$. Then $F_p(s,X_s^p,\Lambda_s^p)$ converges to $F(s,X_s^p,\Lambda_s^p)$, $dt\otimes dP-p.s.$ Moreover

$$F_p(s,X_s^p,\Lambda_s^p)\leq C_1\left(1+|\Lambda_t^p|^2\right)\leq C_1\left(1+\left|\widetilde{\Lambda}_t^p\right|^2\right)$$

for constant C_1.

Proof of the existence

Finally, since the sequence $\left(X^p\right)_{p\geq 1}$ is bounded, the Lebesgue dominated convergence theorem implies that

$$E\left[\int_0^{T\prime}\left(X_s^p - X_s^q\right)\left(F_p\left(s, X_s^p, \Lambda_s^p\right) - F_q\left(s, X_s^q, \Lambda_s^q\right)\right)ds\right] \underset{p,q\to\infty}{\longrightarrow} 0$$

Hence, the sequence $\left(X^p\right)_{p\geq 1}$ converges uniformly to X in L^2 and then X is continuous.

Step 4. Maximal and minimal solutions

Let (X, Λ) and (X^*, Λ^*) two solutions of (1.5). For $p \geq 1$ and $n \geq 1$, let F_p^n be the function defined by :

$$F_p^n(t, x, \lambda) = \sup_{u,v \in \mathbb{R}^2} \{F_p(t, u, v) - n(|u - x| + |v - \lambda|)\}.$$

Since we have $|F_p(t, x, \lambda)| \leq C_1 \left(1 + |\lambda|^2\right)$, the function F_p^n is Lipschitz in (x, λ) and converges toF_p as $n \longrightarrow \infty$. Now let $\left(X_p^n, \Lambda_p^n\right)$ be a solution of the BDSDE associed with $\left(F_p^n, G, \eta\right)$. Since $F_p^n \geq F_p \geq F$, $\;X_p^n \geq X^*$ for all $n, p \geq 1$, then for all $p \geq 1$ we have $\lim_{n\to\infty} X_p^n = X_p$. Therefore $X_p \geq X^*$ and finaly $X \geq X^*$ which implies hat the solution considered is maximal.

The existence of a minimal solution is based on the same arguments. Indeed, let (X_*, Λ_*) another solution of (1.5). For all $p \geq 1$ and $n \geq 1$, let F_p^n be the function defined by

$$F_p^n(t, x, \lambda) = \inf_{u,v \in \mathbb{R}^2} \{F_p(t, u, v) + n(|u - x| + |v - \lambda|)\}.$$

Proof of the existence

Since we have $|F_p(t,x,\lambda)| \leq C_1 \left(1+p^2\right)$, the function F_p^n is Lipschitz in (x,λ) and converges to F_p as $n \to \infty$. Now let $\left(X_p^n, \Lambda_p^n\right)$ be the solution of the BDSDE associed with $\left(F_p^n, G, \eta\right)$. Since $F_p^n \leq F_p, \;\; X_p^n \leq X_*$ for all $n,\ p \geq 1$. therefore for all $p \geq 1$, we have $\lim_{n\to\infty} X_p^n = X_p$. Hence $X_p \leq X_*$ and finaly $X \leq X_*$, *which* implies that the solution considered is minimal.

Proof of theorem (1.4)
We transform equation (1.1) by the exponential function $X_t = e^{2cY_t}$, we are led to solving the following BDSDE :

$$X_t = \eta + \int_t^T F\left(s, X_S, \Lambda_s\right) ds + \int_t^T G\left(s, X_s\right) dB_s - \int_t^T \Lambda_s dW_s. \quad (1.5)$$

Applying Itô's formula to $X_t = e^{2cY_t}$, we get

$$X_t = X_T + \int_t^T 2cX_s f\left(s, Y_s, Z_s\right) ds - \int_t^T 2cX_s g\left(s, Y_s\right) dB_s + \int_t^T 2cX_s Z_s dV$$
$$- \frac{1}{2}\int_t^T 4c^2 X_s \left|g\left(s, Y_s\right)\right|^2 + \frac{1}{2}\int_t^T 4c^2 X_s \left|Z_s\right|^2 ds,$$

where

$$F(s,x,\lambda) = 2cx\left[f\left(s,\frac{\ln x}{2c},\frac{\lambda}{2cx}\right) + c\left|g\left(s,\frac{\ln x}{2c}\right)\right|^2 - \frac{|\lambda|^2}{4cx^2}\right]$$
$$G(s,x) = 2cxg\left(s,\frac{\ln x}{2c}\right),$$

and $X_s = e^{2cY_s}$. This implies in particular that $Y_s = \frac{\ln X_s}{2c}$, $Z_s = \frac{\Lambda_s}{2cX_s}$ and $\eta = e^{2c\xi}$.

It is not difficult to show that the generator $F(s,x,\lambda)$ satisfies the structure condition, and then we are in a position to apply theorem 5, to prove existence of a maximal solution to the BDSDE (1.5) and then the same can be claimed for equation (1.1) with
$Y_s = \frac{\ln X_s}{2c}$, $Z_s = \frac{\Lambda_s}{2cX_s}$.

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Thank you for your attention