Stochastic optimal control problems in Banach spaces

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PLAN

1. SDEs in Banach spaces;
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3. Identification of Z;
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9. Application to stochastic delay equations.

Our framework

SDE with values in $E \subset H$, E Banach, H Hilbert

$$\begin{cases} dX_\tau = [AX_\tau + F(X_\tau)]\, d\tau + G dW_\tau, & \tau \in [t,T] \\ X_t = x, \qquad 0 \le t \le T. \end{cases}$$

Theorem 1: If Hypothesis 1 is satisfied, there exists a unique mild solution $X(\tau,t,x)$, that is an adapted and continuous E-valued process satisfying $\mathbb{P}$-a.s.

$$X_\tau = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-s)A} F(X_s)\, ds + \int_t^\tau e^{(\tau-s)A} G dW_s, \qquad \tau \in [t,T].$$

Proof: See e.g. Da Prato and Zabczyk (1992, 1996).

Hypothesis 1

1. A generates a C_0 semigroup e^{tA}, $t \geq 0$, in E, and there exists $\omega \in \mathbb{R}$ such that $\left\| e^{tA} \right\|_{L(E,E)} \leq e^{\omega t}$, for all $t \geq 0$. e^{tA}, $t \geq 0$ extends to a C_0 semigroup of bounded linear operators in H.

2. $F : E \to E$ continuous and $\exists\ \eta \geq 0$ s.t. $A + F - \eta I$ is dissipative in E.

3. $G \in L(\Xi, H)$ and $Q_\sigma = \displaystyle\int_0^\sigma e^{sA} G G^* e^{sA^*} ds$ is a trace class operator in H.

4. W cylindrical Wiener process in Ξ and $W_A(\tau) = \int_t^\tau e^{(\tau - s)A} G dW_s$ admits an E-continuous version.

Regularity with respect to the initial datum

Let $\mathcal{H}^p([0,T], E) = \left\{\text{predictable processes: } \mathbb{E}\sup_{\tau\in[0,T]} |X_\tau|_E^p < \infty\right\}$.

- $X(\tau, t, x)$ is Lipschitz in x uniformly with respect to τ :

$$\|X(\tau, t, x_1) - X(\tau, t, x_2)\|_E \leq e^{|\eta| T} \|x_1 - x_2\|_E$$

- If F is Gateaux differentiable in E, $X(\tau, t, \cdot)$ is pathwise differentiable.

- Assume that $\exists\ k \geq 0$ s.t. $\|F(x)\|_E \leq c\left(1 + \|x\|_E^k\right) \Rightarrow$

1. $(X(\tau, t, x))_{\tau\in[0,T]} \in \mathcal{H}^p([0,T], E)$

2. the map $x \mapsto (X(\tau, t, x))_{\tau\in[0,T]}$ from E to $\mathcal{H}^p([0,T], E)$ is Gateaux differentiable.

Forward-backward system

$$\begin{cases} dX_\tau = AX_\tau d\tau + F(X_\tau)\, d\tau + G dW_\tau, & \tau \in [t,T] \\ dY_\tau = -\psi(\tau, X_\tau, Z_\tau)\, d\tau + Z_\tau dW_\tau, & \tau \in [t,T] \\ X_t = x, \\ Y_T = \phi(X_T)\, . \end{cases}$$

Hypothesis 2: • For every $\sigma \in [0,T]$, $x \in E$ and $z_1, z_2 \in \Xi^*$

$$|\psi(\sigma, x, z_1) - \psi(\sigma, x, z_2)| \leq L\, |z_1 - z_2|_{\Xi^*}\, .$$

• For every $\sigma \in [0,T]$ $\psi(\sigma, \cdot, \cdot) \in \mathcal{G}^{1,1}(E \times \Xi^*)$ and for every $\sigma \in [0,T]$, $x, h \in E$ and $z \in \Xi^*$

$$|\nabla_x \psi(\sigma, x, z)\, h| \leq L\, \|h\|_E \left(1 + \|x\|_E\right)^m \left(1 + |z|_{\Xi^*}\right).$$

• $\phi \in \mathcal{G}^1(E)$ and lipschitz continuous on E.

• $F \in \mathcal{G}^1(E)$ and $\exists\ k \geq 0$ s.t. $\|F(x)\|_E \leq c\left(1 + \|x\|_E^k\right)$.

Proposition Let hypotheses 1 and 2 hold true. Then the BSDE admits a unique solution $(Y, Z) \in \mathbb{K}_{cont}([0,T])$ and the map $(t,x) \mapsto (Y(\cdot,t,x), Z(\cdot,t,x))$ belongs to $\mathcal{G}^{0,1}([0,T] \times E, \mathbb{K}_{cont}([0,T]))$. The following estimates holds true: for every $p \geq 2$,

$$\left[E \sup_{\tau\in[0,T]} |\nabla_x Y(\tau,t,x)\, h|^p\right]^{1/p} \leq C \|h\|_E \left(1 + \|x\|_E^{(m+1)^2}\right).$$

Corollary Let hypotheses 1 and 2 hold true. Then the function $v(t,x) := Y(t,t,x)$ belongs to $\mathcal{G}^{0,1}([0,T] \times E, \mathbb{R})$ and there exists $C > 0$ such that $|\nabla_x v(t,x)\, h| \leq C \|h\|_E \left(1 + \|x\|_E^{(m+1)^2}\right)$ for all $t \in [0,T]$, $x, h \in E$.

Theorem 2 Let hypotheses 1 and 2 hold true and set $v(t,x) := Y(t,t,x)$, Then, for almost every $s \in [0,T]$, $Z_s\xi = \nabla v(s, X_s)\, G\xi$, $\mathbb{P}$-almost everywhere and for every $\xi \in \Xi_0$.

Main technical result. The argument generalizes the one in Bismut, *Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions*. Z. Wahrsch. Verw. Gebiete (1981).

More in general it holds true for $v \in \mathcal{G}^{0,1}([0,T] \times E, \mathbb{R})$ satisfying

$$v(\tau, X_\tau) = v(T, X_T) + \int_\tau^T \psi_\sigma d\sigma - \int_\tau^T Z_\sigma dW_\sigma, \ \tau \in [t,T].$$

Hypothesis Assume there exists a Banach subspace Ξ_0 dense in Ξ s.t. $G(\Xi_0) \subset E$ and $G : \Xi_0 \longrightarrow E$ is continuous.

Theorem 2: Let X be solution of the SDE, Z and ψ be square integrable processes. Let $v \in G^{0,1}([0,T] \times E)$ s.t. for every $0 \le t \le s \le T$, $|\nabla v(s,x)h| \le c\left(1 + \|x\|_E^j\right)\|h\|_E$, for some integer $j \ge 0$ and for every $x, h \in E$. If

$$v(t,x) + \int_t^T Z_\sigma dW_\sigma = v(T, X_T) + \int_t^T \psi_\sigma d\sigma,$$

then, for almost every $s \in [0,T]$,$Z_s\xi = \nabla v(s, X_s) G\xi$, $\mathbb{P}$-almost everywhere and for every $\xi \in \Xi_0$.

Remark Since Ξ_0 is dense in Ξ, for every $\bar{\xi} \in \Xi$ there exists a sequence $(\xi_n)_n \in \Xi_0$ such that $\xi_n \longrightarrow \bar{\xi}$ in Ξ. For almost every $s \in [0,T]$ and almost surely with respect to the law of X_s, the operator $\nabla v(s,x) G : \Xi_0 \longrightarrow E$ extends to an operator defined in the whole Ξ. So $Z_s = \nabla v(s, X_s) G$, $\mathbb{P}$-almost surely and for almost every $s \in [0,T]$.

Proof:

Let η be a bounded and predictable process with the following form: let $\left[\frac{kT}{2^n}, \frac{(k+1)\,T}{2^n}\right)$, $k = 0, ...2^n - 1$ be a partition $[0, T]$. For $t \in \left[\frac{kT}{2^n}, \frac{(k+1)\,T}{2^n}\right)$

$$\eta_t = \eta^k \left(W_{t_1}, ..., W_{t_{l_k}}\right),\ t \in \left[\frac{kT}{2^n}, \frac{(k+1)\,T}{2^n}\right),\ 0 \leq t_1 \leq ... \leq t_{l_k} \leq \frac{kT}{2^n},\ \eta^k \in C_b^\infty(\mathbb{R}^{l_k}, \mathbb{R}).$$

For $\varsigma \in \Xi_0$, set $\xi_t = \eta_t \varsigma$. Notation: $\xi_t = \xi_t\,(W_\cdot)$, where $(W_\cdot)$ is the trajectory of W up to time t. (see Bismut).

$$v\left(s, X\left(s,t,x\right)\right) + \int_s^T Z_\sigma dW_\sigma = v\left(T, X_T\right) + \int_s^T \psi_\sigma d\sigma$$

and ,for $t \leq s \leq T$,

$$v\left(s, X\left(s,t,x\right)\right) = v\left(t,x\right) + \int_t^s Z_\sigma dW_\sigma - \int_t^s \psi_\sigma d\sigma.$$

$$\mathbb{E}\left[v\left(s, X_s\right)\int_{s-\delta}^s \xi_\sigma^* dW_\sigma\right] = -\mathbb{E}\left[\int_t^{s-\delta}\psi_\sigma d\sigma \int_{s-\delta}^s \xi_\sigma^* dW_\sigma\right]$$
$$-\,\mathbb{E}\left[\int_{s-\delta}^s \psi_\sigma d\sigma \int_{s-\delta}^s \xi_\sigma^* dW_\sigma\right] + \mathbb{E}\left[\int_t^s Z_\sigma dW_\sigma \int_{s-\delta}^s \xi_\sigma^* dW_\sigma\right].$$

$$\Downarrow$$

$$\mathbb{E}\left[Z_s \xi_s\right] = \lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\left[v\left(s, X_s\right)\int_{s-\delta}^s \xi_\sigma^* dW_\sigma\right].$$

Prove that

$$\lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\left[v\left(s,X_s\right)\int_{s-\delta}^{s}\xi_\sigma^* dW_\sigma\right]=\mathbb{E}\left[\nabla v\left(s,X_s\right)G\xi_s\right].$$

Following Bismut,

$$W_\sigma^\varepsilon=W_\sigma-\varepsilon\int_t^\sigma \xi_r\left(W_\cdot^\varepsilon\right)dr,$$

$W_\sigma^\varepsilon=W_\sigma^\varepsilon\left(W_\cdot\right)$. So

$$W_\sigma^\varepsilon=W_\sigma-\varepsilon\int_t^\sigma \xi_r\left(W_\cdot^\varepsilon\left(W_\cdot\right)\right)dr,\ 0\le t\le\sigma\le T.$$

and

$$\frac{d}{d\varepsilon}_{|\varepsilon=0}W_\sigma^\varepsilon=\int_t^\sigma \xi_r\left(W_\cdot\right)dr.$$

Let

$$\frac{dQ_\varepsilon}{d\mathbb{P}} = \mathsf{exp}\left(\varepsilon \int_t^T \xi^*_\sigma \left(W^\varepsilon_\cdot \left(W_\cdot\right)\right) dW_\sigma - \frac{\varepsilon^2}{2} \int_t^T \left|\xi_\sigma \left(W^\varepsilon_\cdot \left(W_\cdot\right)\right)\right|^2 d\sigma\right)$$

By dominated convergence

$$\begin{aligned} \mathbb{E}\left[v\left(s, X_s\right) \int_t^s \xi^*_\sigma dW_\sigma\right] &= \frac{d}{d\varepsilon}_{|\varepsilon=0} \mathbb{E}\left[v\left(s, X_s\right) \mathsf{exp}\left(\varepsilon \int_t^s \xi^*_\sigma dW_\sigma - \frac{\varepsilon^2}{2} \int_t^s \left|\xi_\sigma\right|^2 d\sigma\right)\right] \\ &= \frac{d}{d\varepsilon}_{|\varepsilon=0} \mathbb{E}_{Q_\varepsilon}\left[v\left(s, X_s\right)\right]. \end{aligned}$$

Under Q_ε X solves

$$\begin{cases} dX_\tau = AX_\tau d\tau + F(X_\tau)\,d\tau + G\varepsilon\xi_\tau d\tau + GdW_\tau^\varepsilon, & \tau \in [s-\delta, T] \\ X_{s-\delta} = X(s-\delta, t, x)\,. \end{cases}$$

Under $\mathbb{P}$ X^ε solves

$$\begin{cases} dX_\tau^\varepsilon = AX_\tau^\varepsilon d\tau + F(X_\tau^\varepsilon)\,d\tau + G\varepsilon\xi_\tau d\tau + GdW_\tau, & \tau \in [s-\delta, T] \\ X_t^\varepsilon = X(s-\delta, t, x)\,. \end{cases}$$

$$\Downarrow$$

$$\frac{d}{d\varepsilon}_{|\varepsilon=0}\mathbb{E}_{Q_\varepsilon}[v(s, X_s)] = \frac{d}{d\varepsilon}_{|\varepsilon=0}\mathbb{E}[v(s, X_s^\varepsilon)] = \mathbb{E}\left[\nabla v(s, X_s)\,\dot{X}_s\right]$$

$$\begin{cases} d\dot{X}_\tau = A\dot{X}_\tau d\tau + \nabla F\left(X_\tau\right)\dot{X}_\tau d\tau + G\xi_\tau d\tau, & \tau \in [s-\delta, T] \\ \dot{X}_{s-\delta} = 0. \end{cases}$$

claim: $\dot{X}_\tau = \int_t^\tau \nabla X\left(\tau, \sigma, X\left(\sigma, t, x\right)\right) G\xi_\sigma d\sigma$

$$\Downarrow$$

$$\begin{aligned} \mathbb{E}\left[Z_s\xi_s\right] &= \lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\left[\nabla v\left(s, X_s\right)\int_{s-\delta}^{s}\xi_\sigma^* dW_\sigma\right] \\ &= \lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\left[\nabla v\left(s, X_s\right)\int_{s-\delta}^{s}\nabla X\left(s, \sigma, X\left(\sigma, t, x\right)\right) G\xi_\sigma d\sigma\right] \\ &= \mathbb{E}\left[\nabla v\left(s, X_s\right)\nabla X\left(s, s, X\left(s, t, x\right)\right) G\xi_s\right] \\ &= \mathbb{E}\left[\nabla v\left(s, X_s\right) G\xi_s\right]. \end{aligned}$$

The optimal control problem: weak formulation

controlled SDE

$$\begin{cases} dX_\tau^u = \left[AX_\tau^u + F\left(X_\tau^u\right) + GR\left(\tau, X_\tau^u, u_\tau\right)\right] d\tau + GdW_\tau \\ X_t^u = x \in E, \qquad\qquad \tau \in [t, T]. \end{cases}$$

$$u \in L^2_{\mathcal{P}}\left(\Omega \times [0, T], U\right).$$

Cost functional and value function

$\mathbb{A} = (\Omega, \mathcal{F}, \mathcal{F}_\tau, \mathbb{P}, W, u, X^u)$ admissible control system (a.c.s.).

$$J(t, x, \mathbb{A}) = \mathbb{E} \int_t^T g\left(s, X_s^u, u_s\right) ds + \mathbb{E}\phi\left(X_T^u\right),$$
$$J^*(t, x) = \inf_{\mathbb{A}} J(t, x, \mathbb{A}).$$

Hypothesis 3: $R : [0,T] \times H \times U \longrightarrow \Xi$ measurable. $\forall \tau \in [0,T]$,

$$|R(\tau, x, u)| \leq K_R.$$

$\phi \in \mathcal{G}^1(E)$ and lipschitz continuous on E. $g : [0,T] \times E \times U \longrightarrow \mathbb{R}$, continuous. $\exists\ K > 0$ s.t. for $j \geq 0$, for every $x \in E$

$$|g(\tau, x, u)| \leq K \left(1 + \|x\|_E^j\right).$$

Weak formulation of the optimal control problem (see e.g. Fleming-Soner 1993): find an a.c.s. $\overline{\mathbb{A}}$ s.t. $J\left(t, x, \overline{\mathbb{A}}\right) \leq J(t, x, \mathbb{A})$ for every a.c.s. $\mathbb{A}$. Then $\overline{\mathbb{A}}$ is optimal.

Hamiltonian function: for every $\tau \in [0,T]$, $x \in E$, $z \in \Xi^*$,

$$\psi(\tau,x,z) = \inf\left\{g(\tau,x,u) + zR(\tau,x,u) : u \in \mathcal{U}\right\}.$$

We assume that $\psi(\sigma,\cdot,\cdot) \in \mathcal{G}^{1,1}(E \times \Xi^*)$ and

$$|\nabla_x \psi(\sigma,x,z)\,h| \leq L\,\|h\|_E \left(1+\|x\|_E\right)^m \left(1+|z|_{\Xi^*}\right).$$

Fundamental relation in terms of BSDE

$$\begin{aligned} v(t,x) &= J(t,x,\mathbb{A}) \\ &\quad + \mathbb{E}\int_t^T \left[\psi(\sigma,X_\sigma^u,Z_\sigma^u) - Z_\sigma^u R(\sigma,X_\sigma^u,u_\sigma) - g(\sigma,X_\sigma^u,u_\sigma)\right] d\sigma. \end{aligned}$$

Moreover $Z_\sigma^u = \nabla v(\sigma,X_\sigma^u)\,G$.

u is optimal iff $u_\tau \in \Gamma\left(\tau, X_\tau^u, \nabla v\left(\tau, X_\tau^u\right) G\right)$, $\mathbb{P}$-a.s. for a.a. $\tau \in [t, T]$.

Closed loop equation: for $\tau \in [t, T]$

$$\begin{cases} d\overline{X}_\tau = \left[A\overline{X}_\tau + F\left(\overline{X}_\tau\right) + GR\left(\tau, \overline{X}_\tau, \Gamma\left(\tau, \overline{X}_\tau, \nabla v\left(\tau, \overline{X}_\tau\right) G\right)\right)\right] d\tau + G dW_\tau, \\ \overline{X}_t = x. \end{cases}$$

Theorem 3: Under hypotheses 1, 2, 3 there exists an optimal a.c.s. and the optimal trajectory is solution of the closed loop equation.

Hamilton Jacobi Bellman equation on a Banach space E.

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = -\mathcal{A}v\,(t,x) - \psi\,(t,x,\nabla v\,(t,x)\,G)\,, & t\in[0,T]\,,\ x\in E \\ v(T,x) = \phi\,(x)\,, \end{cases}$$

where

$$\mathcal{A}f\,(x) = \frac{1}{2}Trace_H\left(GG^*\nabla^2 f\,(x)\right) + \langle Ax, \nabla f\,(x)\rangle_{E,E^*} + \langle F\,(x)\,, \nabla f\,(x)\rangle_{E,E^*}.$$

Let $P_{t,\tau}\,[\phi]\,(x) = E\phi\,(X\,(\tau,t,x))$. v mild solution

$$v(t,x) = P_{t,T}\,[\phi]\,(x) + \int_t^T P_{t,\tau}\,[\psi(\tau,\cdot,\nabla v\,(\tau,\cdot)\,G]\,(x)\,d\tau, \quad t\in[0,T]\,,\ x\in E.$$

Theorem 4: With hypothesis 1 and 2, there exists a unique mild solution $v(t,x) = Y(t,t,x)$, where (Y,Z) is the unique solution of the BSDE.

$$\begin{cases} dX_\tau = [AX_\tau + F(X_\tau)]\, d\tau + G dW_\tau, & \tau \in [t,T] \\ X_t = x, \qquad 0 \le t \le T. \end{cases}$$

Do not impose any growth conditions on F. If F is Gateaux differentiable in E, $X(\tau, t, \cdot)$ is pathwise differentiable. In general, $X \notin \mathcal{H}^p([0,T], E)$.

ψ and ϕ in the BSDE

$$\begin{cases} dY_\tau = -\psi(\tau, X_\tau, Z_\tau)\, d\tau + Z_\tau dW_\tau, & \tau \in [t,T] \\ \\ Y_T = \phi(X_T). \end{cases}$$

must be taken bounded.

- Recover differentiability of Y and Z w.r.to x.
- Identification of Z.

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Stochastic controlled semilinear heat equations on $[0,1]$.

$$\begin{cases} d_\tau X^u(\tau,\xi) = \left[\frac{\partial^2}{\partial\xi^2} X^u(\tau,\xi) + f(\tau,\xi,X^u(\tau,\xi)) + \chi_{\mathcal{O}}(\xi) u(\tau,\xi)\right] d\tau + \chi_{\mathcal{O}}(\xi)\,\dot{W}(\tau,\xi)\,d\tau, \\ X^u(\tau,0) = X^u(\tau,1) = 0, \\ X^u(t,\xi) = x_0(\xi), \end{cases}$$

$\mathcal{O} \subset [0,1]$, namely $\mathcal{O} = [a,b]$.

cost functional

$$J(t,x,(W,U,x^u)) = \mathbb{E}\int_t^T \int_0^1 l(s,\xi,X^u(s,\xi),u)\,\mu(d\xi)\,ds + \mathbb{E}\int_0^1 k(\xi,X^u(T,\xi))\,\mu(d\xi)$$

Abstract formulation

$$H = \Xi = L^2([0,1]),\ E = C([0,1]).$$

$$\begin{cases} dX_\tau^u = \left[AX_\tau^u + F(\tau, X_\tau^u)\right] d\tau + GRu_\tau d\tau + GdW_\tau & \tau \in [t,T] \\ X_t^u = x_0, \end{cases}$$

where

$$F(\tau, x)(\xi) = f(\tau, \xi, x(\xi)), \qquad (Gz)(\xi) = \chi_{\mathcal{O}}(\xi)\, z(\xi)$$

$$g(\tau, x, u)(\xi) = \int_0^1 l(s, \xi, x(\xi), u)\, \mu(d\xi), \quad \phi(x)(\xi) = \int_0^1 k(\xi, x(\xi))\, \mu(d\xi).$$

Hypothesis

- $f \in C([0,T] \times [0,1] \times \mathbb{R}, \mathbb{R})$, $\forall\ \tau \in [0,T]$, $\xi \in [0,1]$, $f(\tau, \xi, \cdot) \in C^1(\mathbb{R})$. $\forall$ $\tau \in [0,T]$, $\xi \in [0,1]$, $x \in \mathbb{R}$ the map $x \longmapsto f(\tau, \xi, x)$ is decreasing
- $l \in C_b([0,T] \times [0,1] \times \mathbb{R} \times \mathcal{U}, \mathbb{R})$.
- $k \in C_b([0,1] \times \mathbb{R}, \mathbb{R})$ and $k(\xi, \cdot) \in C_b^1(\mathbb{R})$.
- $x_0 \in C([0,1])$.

$\Xi_0 = \{f \in C([0,1]) : f(a) = f(b) = 0\}$, where $\mathcal{O} = [a,b]$. So $G : \Xi_0 \longmapsto E = C([0,1])$.

An example of a cost: $\mu = \sum_{i=1}^{N} \delta_{\xi_i}, \ \xi_1, ..., \xi_N \in [0,1]$.

Hamiltonian

$$\psi(t,x,z) = \inf_{u \in \mathcal{U}} \left\{ \sum_{i=1}^{N} l(t, \xi_i, x(\xi_i), u) + \int_0^1 z(\xi) r(\xi) u(\xi) d\xi \right\}.$$

Let $\overline{\psi} : [0,T] \times \mathbb{R}^N \times \Xi^* \to \mathbb{R}$

$$\overline{\psi}(t, y_1, ..., y_N, z) = \inf_{u \in \mathcal{U}} \left\{ \sum_{i=1}^{N} l(t, \xi_i, y_i, u) + \int_0^1 z(\xi) r(\xi) u(\xi) d\xi \right\}.$$

If $\sum_{i=1}^{N} l(t, \xi_i, y_i, u) = \sum_{i=1}^{N} l(t, \xi_i, y_i) + \displaystyle\int_0^1 \frac{u^2(\xi)}{2} d\xi$, then $\overline{\psi}(t, \cdot, ..., \cdot, \cdot) : \mathbb{R}^N \times \Xi^* \to \mathbb{R}$ is differentiable with bounded derivatives.

Lemma: $W_A(\tau)$ admits a continuous version in $C([0,T],E)$.

Theorem: Under the previous assumptions, equation

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = -\mathcal{A}_t v(t,x) - \psi(t,x,\nabla v(t,x)G), & t \in [0,T], \ x \in H, \\ u(T,x) = \phi(x), \end{cases}$$

has a unique mild solution v and for all admissible control systems (W,u,X^u), $J(t,x,(W,u,X^u)) \geq v(t,x)$. Moreover there exists an optimal a.c.s. and the optimal trajectory is solution of the closed loop equation.

Stochastic delay equations

$$\begin{cases} dz^u(\tau) = \left[\int_{-r}^{0} d\eta(\theta)\, z^u(\tau+\theta)\right] d\tau + u(\tau)\, d\tau + dW(\tau), & \tau \in [0,T] \\ z^u(0) = h_0 \in \mathbb{R}, \\ z^u(\theta) = h_1(\theta), \qquad \theta \in [-r,0], & h_1 \in L^p([-r,0],\mathbb{R}). \end{cases}$$

Cost functional

$$J(t,h_0,h_1,u) = \mathbb{E}\int_{-r}^{0} z^u(T+\theta)\, y(\theta)\, d\theta,$$

where $y \in L^q([-r,0],\mathbb{R})$.

Define $E = \mathbb{R} \oplus L^p\left([-r, 0], \mathbb{R}\right)$ and A by

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} \in E, h_1 \in W^{1,p}\left([-r,0], \mathbb{R}\right), h_1(0) = h_0 \right\},$$

$$Ah = A \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} \int_{-r}^{0} a\,(d\theta)\, h_1(\theta) \\ dh_1/d\theta \end{pmatrix}.$$

Abstract formulation in E: $X_\tau = \begin{pmatrix} z(\tau) \\ z_\tau \end{pmatrix}$, where $z_\tau(\theta) = z(\tau + \theta)$.

$$\begin{cases} dX_\tau = AX_\tau d\tau + Gu_\tau d\tau + GdW_\tau, & \tau \in [0, T] \\ X_0 = h. \end{cases}$$

where

$$G : \mathbb{R} \longrightarrow E,\ G = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Set for $j = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \in E$, $\phi(j) = \int_{-r}^{0} j_2(\theta)\, y(\theta)\, d\theta$. Abstract cost functional

$$J(t, h, u) = \mathbb{E}\phi(X_T^u).$$

Theorem 5: There exists an optimal a.c.s. and the optimal trajectory is solution of the closed loop equation.

Application to finance

$$\begin{cases} dS_\tau = rS_\tau dW_\tau, & \tau \in [0,T] \\ S_0 = s_0. \end{cases}$$

exotic option with contingent claim

$$\varphi\left(S_T\left(\cdot\right)\right), \quad \text{where } S_T\left(\theta\right) = S_{T+\theta},\ \theta \in \left[-T, 0\right].$$

We can treat contingent claims such as

$$\varphi\left(S_T\left(\cdot\right)\right) = \int_{-T}^{0} k\left(\theta\right) S_T\left(\theta\right) d\theta,$$

obtaining an infinite dimensional analogue of Black-Scholes