Journées de Probabilités 2007
La Londe, 10-14 septembre 2007

Principe de Maximum et Théorème de Comparison pour les solutions d'EDPS quasi-linéaires SPDE's

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Problem :

We study the following stochastic partial differential equation (in short SPDE) for a real -valued random field $u_t(x) := u(t,x)$,

$$du_t(x) = Lu_t(x)\,dt + f_t(x, u_t(x), \nabla u_t(x))\,dt + \sum_{i=1}^{d} \partial_i g_{i,t}(x, u_t(x), \nabla u_t(x))\,dt$$

$$+ \sum_{j=1}^{d_1} h_{j,t}(x, u_t(x), \nabla u_t(x))\,dB_t^j \qquad (1)$$

with a given initial condition $u_0 = \xi$, where L is a symmetric second order differential operator defined in some bounded open domain $\mathcal{O} \subset \mathbb{R}^d$ and $f, g_i, i = 1, ..., d, h_j, j = 1, ..., d_1$ are nonlinear random functions.

We study :

- the maximum principle for the SPDE (E)

- comparison theorem.

- existence and uniqueness of the solution of the stochastic Burger equation.

The maximum principle for quasilinear parabolic equations **(the deterministic case : $h = 0$)** was proved by Aronson -Serrin (1967) in the following form :

Theorem : Let u be a weak solution of a quasilinear parabolic equation of the form

$$\partial_t u = div\mathcal{A}\left(t, x, u, \nabla u\right) + \mathcal{B}\left(t, x, u, \nabla u\right)$$

in the bounded cylinder $]0, T[\times\mathcal{O} \subset \mathbb{R}^{d+1}$.

If $u \leq M$ on the parabolic boundary $\{[0, T[\times\partial\mathcal{O}\} \cup \{\{0\} \times \mathcal{O}\}$, then one has

$$u \leq M + Ck\left(\mathcal{A}, \mathcal{B}\right),$$

where C depends only on T, the volume of $\mathcal{O}$ and the structure of the equation, while $k\left(\mathcal{A}, \mathcal{B}\right)$ is directly expressed in terms of some quantities related to the coefficients $\mathcal{A}$ and $\mathcal{B}$.

The method of proof was based on Moser's iteration scheme adapted to the nonlinear case. This method was further adapted to the stochastic framework in Denis, M. and Stoica (2005), obtaining some L^p a priori estimates for the uniform norm of the solution of the stochastic quasilinear parabolic equation.

We prove the stochastic version of the maximum principle of Aronson -Serrin :

Theorem :

Let $p \geq 2$ and u be a solution of (1) in the weak sense. Assume that $u \leq M$ on the parabolic boundary $\{[0,T[\times\partial\mathcal{O}\} \cup \{\{0\} \times \mathcal{O}\}$, then for all $t \in [0,T]$:

$$E \left\|(u-M)^+\right\|^p_{\infty,\infty;t} \leq k\,(p,t)\,E\left(\|(\xi-M)^+\|^p_\infty + \left\|(f^{0,M})^+\right\|^{*p}_{\theta,t} + \left\||g^{0,M}|^2\right\|^{*p/2}_{\theta;t} + \left\||h^{0,M}|^2\right\|^{*p/2}_{\theta;t}\right)$$

where :

$f^{0,M}(t,x) = f(t,x,M,0)$, $g^{0,M}(t,x) = g(t,x,M,0)$, $h^{0,M}(t,x) = h(t,x,M,0)$ and

k is a function which only depends on the structure constants of the SPDE,

$\|\cdot\|_{\infty,\infty;t}$ is the uniform norm on $[0,t]\times\mathcal{O}$ and $\|\cdot\|^*_{\theta;t}$ is a certain norm which is precisely defined below.

Hypothesis and definitions :

◇ $\mathcal{O} \subset \mathbb{R}^d$ open bounded set.

◇ $(B_t)_t$ d_1-dimensional BM defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$,

◇ $A := -L := -\sum \partial_i(a^{i,j}\partial_j)$: symmetric second order differential operator,

◇ $a := (a^{ij})_{i,j}$ is a measurable and symmetric matrix and satisfies uniform ellipticity :

$$\lambda|\zeta|^2 \leq \sum_{i,j=1}^{d} a^{i,j}(x)\zeta^i \zeta^j \leq \Lambda|\zeta|^2, \quad \forall x \in \mathcal{O},\ \zeta \in \mathbb{R}^d$$

where λ and Λ are positive constants.

◇ $\xi \in L^2(\Omega \times \mathcal{O})$.

◇ $T > 0$.

We are given predictable functions :

$$\begin{aligned} f &: \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}\ , \\ h &: \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d_1} \\ g &= (\bar{g}_1, ..., \bar{g}_d)\ : \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d. \end{aligned}$$

such that :

1. $|f(t,\omega,x,y,z) - f(t,\omega,x,y',z')| \leq C(|y-y'| + |z-z'|)$

2. $\left(\sum_{j=1}^{d_1} |h_j(t,\omega,x,y,z) - h_j(t,\omega,x,y',z')|^2 \right)^{\frac{1}{2}} \leq C\,|y-y'| + \beta\,|z-z'|,$

3. $\left(\sum_{i=1}^{d} |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y',z')|^2 \right)^{\frac{1}{2}} \leq C|y-y'| + \alpha\,|z-z'|,$

where C, α, β are non negative constants.

Contraction hypothesis :

$$\alpha + \frac{1}{2}\beta^2 < \lambda.$$

Weak solutions of SPDE's :

- $\mathcal{H}_0$: set of $H^1_0(\mathcal{O})$-valued predictable processes u s.t.

$$\|u\|_{E,T} := \left(E \sup_{0\le t\le T} \|u_t\|^2 + \int_0^T E\,\mathcal{E}\left(u_t, u_t\right) dt \right)^{1/2} < \infty \ .$$

where $\mathcal{E}$ is the energy (Dirichlet form associated to the linear operator A) :

$$\mathcal{E}(u,v) := \int_{\mathcal{O}} \sum_{i,j=1}^{d} a^{i,j} \partial_i u \, \partial_j v \, dx, \quad \forall u \in H^1_{loc}(\mathcal{O}), \ \forall v \in H^1_0(\mathcal{O}).$$

- $\mathcal{H}_{loc}$: set of $H^1_{loc}(\mathcal{O})$-valued predictable processes such that for any compact subset K in $\mathcal{O}$:

$$\|u\|_{E,K,T} := \left(E \sup_{0\le t\le T} \int_K u_t(x)^2 \, dx + E \int_0^T \int_K |\nabla u_t(x)|^2 \, dx dt \right)^{1/2} < \infty \, .$$

Definition :

$u \in \mathcal{H}_{loc}$ is a weak solution of (E), with initial condition $u_0 = \xi$, if for each test function $\varphi \in \mathcal{D} := \mathcal{C}_c^\infty([0,T)) \otimes C_c^2(\mathcal{O})$.

$$\int_0^T [(u_s, \partial_s \varphi) - \mathcal{E}(u_s, \varphi_s) + (f(s, u_s, \nabla u_s), \varphi_s) - (g_i(s, u_s, \nabla u_s), \partial_i \varphi_s)] ds$$
$$+ \int_0^T (h_j(u_s, \nabla u_s), \varphi_s) dB_s^j + (\xi, \varphi_0) = 0.$$

where $(\ ,\ )$ is the inner product in $L^2(\mathcal{O})$.

We denote by $\mathcal{U}_{loc}(\xi, f, g, h)$ the set of such solution.

If $u \in \mathcal{H}_0$ is a weak solution, we say that it solves (E) with zero Dirichlet condition on $\partial\mathcal{O}$ and we denote $u = \mathcal{U}_0(\xi, f, g, h)$.

Functional spaces :

• We shall use the notation

$$(u, v) = \int_{\mathcal{O}} u(x)v(x)\,dx,$$

where u, v are measurable functions defined in $\mathcal{O}$ and $uv \in L^1(\mathcal{O})$.

• $H_0^1(\mathcal{O})$ the Hilbert space : the first order Sobolev space of functions vanishing at the boundary, Its natural scalar product and norm are

$$(u, v)_{H_0^1(\mathcal{O})} = (u, v) + \int_{\mathcal{O}} \sum_{i=1}^{d} (\partial_i u(x))(\partial_i v(x))\,dx,\ \|u\|_{H_0^1(\mathcal{O})} = \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}}.$$

• $H_{loc}^1(\mathcal{O})$ the space of functions which are locally square integrable in $\mathcal{O}$ and which admit first order derivatives that are also locally square integrable.

• For each $t > 0$ and for all real numbers p, $q \geq 1$, we denote by $L^{p,q}([0,t] \times \mathcal{O})$ the space of (classes of) measurable functions $u : [0,t] \times \mathcal{O} \longrightarrow \mathbb{R}$ such that

$$\|u\|_{p,q;\,t} := \left(\int_0^t \left(\int_{\mathcal{O}} |u(t,x)|^p\,dx\right)^{q/p} dt\right)^{1/q}$$

is finite. The limiting cases with p or q taking the value ∞ are also considered with the use of the essential sup norm. We identify this space, in an obvious way, with the space $L^q([0,t]; L^p(\mathcal{O}))$, consisting of all measurable functions $u : [0,t] \to L^p(\mathcal{O})$ such that $\int_0^t \|u_s\|_p^q \, ds < \infty$. This identification implies that $\left(\int_0^t \|u_s\|_p^q \, ds \right)^{\frac{1}{q}} = \|u\|_{p,q;t}$.

• The space of measurable functions $u : \mathbb{R}_+ \to L^2(\mathcal{O})$ such that $\|u\|_{2,2;t} < \infty$, for each $t \geq 0$, is denoted by $L^2_{loc}(\mathbb{R}_+; L^2(\mathcal{O}))$.

• Similarly, the space $L^2_{loc}(\mathbb{R}_+; H^1_0(\mathcal{O}))$ consists of all measurable functions $u : \mathbb{R}_+ \to H^1_0(\mathcal{O})$ such that

$$\|u\|_{2,2;t} + \|\nabla u\|_{2,2;t} < \infty,$$

for any $t \geq 0$.

• Next we are going to introduce some other spaces of functions of interest which have already been used in Aronson and Serrin :

Let $(p_1, q_1), (p_2, q_2) \in [1, \infty]^2$ be fixed and set

$$I = I(p_1, q_1, p_2, q_2) := \left\{ (p, q) \in [1, \infty]^2 \, / \, \exists \, \rho \in [0,1] \, s.t. \, \frac{1}{p} = \rho \frac{1}{p_1} + (1-\rho) \frac{1}{p_2}, \frac{1}{q} = \rho \frac{1}{q_1} + (1-\rho) \frac{1}{q_2} \right\}.$$

This means that the set of inverse pairs $\left(\frac{1}{p}, \frac{1}{q}\right)$, (p, q) belonging to I, is a segment contained in the square $[0,1]^2$, with the extremities $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$ and $\left(\frac{1}{p_2}, \frac{1}{q_2}\right)$.

• There are two spaces of interest associated to I. One is the intersection space

$$L_{I;t} = \bigcap_{(p,q)\in I} L^{p,q}\left([0,t] \times \mathcal{O}\right).$$

Hölder's inequality lead to the following inclusion :

$$L^{p_1,q_1}\left([0,t] \times \mathcal{O}\right) \cap L^{p_2,q_2}\left([0,t] \times \mathcal{O}\right) \subset L^{p,q}\left([0,t] \times \mathcal{O}\right),$$

for each $(p, q) \in I$, and the inequality

$$\|u\|_{p,q;t} \leq \|u\|_{p_1,q_1;t} \vee \|u\|_{p_2,q_2;t}$$

for any $u \in L^{p_1,q_1}\left([0,t] \times \mathcal{O}\right) \cap L^{p_2,q_2}\left([0,t] \times \mathcal{O}\right)$.

• Moreover, by Hölder's inequality, it follows that one has

$$\int_0^t \int_{\mathcal{O}} u(s,x)\, v(s,x)\, dx ds \leq \|u\|_{I;t} \|v\|^{I';t}, \tag{2}$$

for any $u \in L_{I;t}$ and $v \in L^{I';t}$. This inequality shows that the scalar product of $L^2\left([0,t] \times \mathcal{O}\right)$ extends to a duality relation for the spaces $L_{I;t}$ and $L^{I';t}$.

• Now let us recall that the Sobolev inequality states that

$$\|u\|_{2^*} \leq c_S \|\nabla u\|_2,$$

for each $u \in H_0^1(\mathcal{O})$, where $c_S > 0$ is a constant that depends on the dimension and $2^* = \frac{2d}{d-2}$ if $d > 2$, while 2^* may be any number in $]2, \infty[$ if $d = 2$ and $2^* = \infty$ if $d = 1$.

• Therefore one has

$$\|u\|_{2^*,2;t} \leq c_S \|\nabla u\|_{2,2;t},$$

for each $t \geq 0$ and each $u \in L^2_{loc}(\mathbb{R}_+; H_0^1(\mathcal{O}))$.

• if $u \in L^{\infty}_{loc}\left(\mathbb{R}_+; L^2\left(\mathcal{O}\right)\right) \bigcap L^2_{loc}\left(\mathbb{R}_+; H^1_0\left(\mathcal{O}\right)\right)$, one has

$$\left\|u\right\|_{2,\infty;t} \vee \left\|u\right\|_{2^*,2;t} \leq c_1 \left(\left\|u\right\|^2_{2,\infty;t} + \left\|\nabla u\right\|^2_{2,2;t}\right)^{\frac{1}{2}},$$

with $c_1 = c_S \vee 1$.

• One particular case of interest for us in relation with this inequality is when $p_1 = 2, q_1 = \infty$ and $p_2 = 2^*, q_2 = 2$. If $I = I\left(2,\infty,2^*,2\right)$, then the corresponding set of associated conjugate numbers is $I' = I'\left(2,\infty,2^*,2\right) = I\left(2,1,\frac{2^*}{2^*-1},2\right)$, where for $d = 1$ we make the convention that $\frac{2^*}{2^*-1} = 1$.

• In this particular case we shall use the notation $L_{\#;t} := L_{I;t}$ and $L^*_{\#;t} := L^{I';t}$ and the respective norms will be denoted by

$$\left\|u\right\|_{\#;t} := \left\|u\right\|_{I;t} = \left\|u\right\|_{2,\infty;t} \vee \left\|u\right\|_{2^*,2;t}, \ \left\|u\right\|^*_{\#;t} := \left\|u\right\|^{I';t}.$$

Thus we may write

$$\left\|u\right\|_{\#;t} \leq c_1 \left(\left\|u\right\|^2_{2,\infty;t} + \left\|\nabla u\right\|^2_{2,2;t}\right)^{\frac{1}{2}}, \tag{3}$$

for any $u \in L^{\infty}_{loc}\left(\mathbb{R}_+; L^2\left(\mathcal{O}\right)\right) \bigcap L^2_{loc}\left(\mathbb{R}_+; H^1_0\left(\mathcal{O}\right)\right)$ and $t \geq 0$ and the duality inequality becomes

$$\int_0^t \int_{\mathcal{O}} u\left(s,x\right) v\left(s,x\right) dx ds \leq \left\|u\right\|_{\#;t} \left\|v\right\|^*_{\#;t},$$

for any $u \in L_{\#;t}$ and $v \in L^*_{\#;t}$.

Hypotheses on initial conditions :

For a certain $p \geq 2$:

- $\xi \in L^p(\Omega;\, L^\infty(\mathcal{O}))$.

- There exists $\theta \in (0,1)$ such that $\|f^0\|^*_{\theta;T}$, $\left(\||g^0|^2\|^*_{\theta;T}\right)^{1/2}$, $\left(\||h^0|^2\|^*_{\theta;T}\right)^{1/2}$ are in $L^p\,(\Omega, P)$.

$\diamond$ $f^0 := f(.,.,.,0,0) \in L^2([0,\, T] \times \Omega \times \mathcal{O})$ (resp. g^0 and h^0).

- **Assumption (HD)** *local integrability conditions on* f^0, g^0 *and* h^0 :

$$E \int_0^t \int_K (|f_t^0(x)| + |g_t^0(x)|^2 + |h_t^0|^2\,)dxdt < \infty$$

for any compact set $K \subset \mathcal{O}$, and for any $t \geq 0$.

- **Assumption (HI)** *local integrability condition on the initial condition* :

$$E \int_K |\xi(x)|^2 dx < \infty$$

for any compact set $K \subset \mathcal{O}$.

• **Assumption (HD#)**

$$E\left(\left(\|f^0\|^*_{\#;t}\right)^2 + \|g^0\|^2_{2,2;t} + \|h^0\|^2_{2,2;t}\right) < \infty,$$

for each $t \geq 0$.

Sometimes we shall consider the following stronger forms of these conditions :

• **Assumption (HD2)**

$$E\left(\|f^0\|^2_{2,2;t} + \|g^0\|^2_{2,2;t} + \|h^0\|^2_{2,2;t}\right) < \infty,$$

for each $t \geq 0$.

• **Assumption (HI2)** *integrability condition on the initial condition* :

$$E\|\xi\|^2 < \infty.$$

I. Solution of (E) with Zero Dirichlet condition

• L. Denis, A. M. and L. Stoica : *L^p estimates for the uniform norm of solutions of quasilinear SPDE's*. Prob. Theor. Related Fields (2005).

Theorem 1 *Equation (E) admits a unique solution, u in $\mathcal{H}_0$. This solution has $L^2(\mathcal{O})$-continuous trajectories and it satisfies the following estimate $\forall t \in [0,T]$:*

$$E\,\|u\|^p_{\infty,\infty;t} \leq k\,(p,t)\,E\left(\|\xi\|^p_\infty + \left\|f^0\right\|^{*p}_{\theta,t} + \left\||g^0|^2\right\|^{*p/2}_{\theta;t} + \left\||h^0|^2\right\|^{*p/2}_{\theta;t}\right),$$

where k is a function which only depends on C, α and β.

Main tools for the proof :

- Ito's formula

Lemma 2 : *Let* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *be* C^2 *with bounded derivatives. Then a.s., for all* $t \geq 0$

$$\int_{\mathcal{O}} \varphi(u_t(x))\, dx \; + \int_0^t \mathcal{E}(\varphi'(u_s),\, u_s)\, ds = \int_{\mathcal{O}} \varphi(\xi)\, dx + \int_0^t (\varphi'(u_s), f_s)\, ds$$
$$-\sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \varphi''(u_s(x)) \partial_i u_s(x)\, g_i(s,x)\, dx\, ds + \sum_{j=1}^{d_1} \int_0^t (\varphi'(u_s),\, h_j(s))\, dB_s^j$$
$$+\frac{1}{2} \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi''(u_s(x)) h_j^2(s,x)\, dx\, ds\,.$$

- As a consequence (!)

Lemma 3 *For all $l \geq 2$, P-almost surely, for all $t \geq 0$*

$$\int_{\mathcal{O}} |u_t(x)|^l \, dx + \int_0^t \mathcal{E}\left(l\,(u_s)^{l-1}\, sgn(u_s),\, u_s\right) ds = \int_{\mathcal{O}} |\xi(x)|^l \, dx$$

$$+ \; l \int_0^t \int_{\mathcal{O}} sgn(u_s)|u_s(x)|^{l-1} f(s,x,u_s,\nabla u_s)\, dxds$$

$$- \;\; l(l-1) \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} |u_s(x)|^{l-2} \partial_i u_s(x)\, g_i(s,x,u_s,\nabla u_s)\, dx\, ds$$

$$+ \, l \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} sgn(u_s)|u_t(x)|^{l-1} h_j(s,x,u_s,\nabla u_s)\, dx dB_s^j$$

$$+ \frac{l(l-1)}{2} \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} |u_t(x)|^{l-2} h_j^2(s,x,u_s,\nabla u_s)\, dx\, ds\,.$$

where $\mathcal{E}\left(l\,(u_s)^{l-1}\, sgn(u_s),\, u_s\right) = l(l-1) \sum_{i,j=1}^{d} \int_{\mathcal{O}} |u_s(x)|^{l-2} a^{ij}(x)\, \partial_i u_s(x)\, \partial_j u_s(x)\, dx.$

- Sobolev's inequality.
- Estimates of the stochastic part thanks to the theory of domination of processes.
- Aronson-Serrin's Iteration (or Moser's scheme)

A comparison Theorem :

We keep same hypotheses.

We are given another function $\xi' \in L^p(\Omega, L^\infty(\mathcal{O}))$ and another coefficient $\bar{f}$ which satisfy the same assumptions as f. We still consider $u = \mathcal{U}_0(\xi, f, g, h)$ and we set $v = \mathcal{U}_0(\xi', \bar{f}, g, h)$.

Theorem 4 *Assume* $\xi \geq \xi'$ $dP \otimes dx$*-a.e. and*

$$f(t, w, x, u_t(x), \nabla u_t(x)) \geq \bar{f}(t, w, x, u_t(x), \nabla u_t(x)), \quad dt \otimes dP \otimes dx\, a.e$$

Then, for all $t \in [0, T]$,

$$u_t \geq v_t; \quad dP \otimes dx - a.e.$$

Idea of the proof : Apply Ito's formula to $|(u_t - v_t)^+|^2$ and Gronwall's Lemma.

II. Local Solution of (E) with non Zero Boundary condition

• L. Denis, A. M. and L. Stoica : *Maximum principle and Comparison Theorem for Quasilinear SPDE's*. Preprint (2007), submitted.

• We consider the parabolic boundary : $\Gamma = \{\partial\mathcal{O} \times [0,T]\} \bigcup \{\mathcal{O} \times (t=0)\}$.
• M, an $\mathbb{R}$-valued process given by

$$\forall t \geq 0,\ M_t = m + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dB_s,$$

where b , σ are predictable processes such that :

$$E\left(\int_0^T |b_s|^{\frac{1}{1-\theta}} ds\right)^p < +\infty \quad \text{and} \quad E\left(\int_0^T |\sigma_s|^{\frac{2}{1-\theta}} ds\right)^p < +\infty.$$

In particular, these conditions imply that $E(\|M\|^{*p}_{\theta,T}) < +\infty$.

• Let us now precise the sense in which a solution is dominated on the lateral boundary : Assume that v belongs to $H^1_{loc}(\mathcal{O}')$ where $\mathcal{O}'$ is a larger open set such that $\overline{\mathcal{O}} \subset \mathcal{O}'$. Then the condition $v^+_{|\mathcal{O}} \in H^1_0(\mathcal{O})$ expresses the boundary relation $v \leq 0$ on $\partial\mathcal{O}$.

Similarly,

Definition 5 *if a process* u *belongs to* $\mathcal{H}_{loc}(\mathcal{O}')$*, then the condition* $u^+_{|\mathcal{O}} \in \mathcal{H}_0$ *ensures the inequality* $u \leq 0$ *on the lateral boundary* $\{[0,\infty[\times\partial\mathcal{O}\}$.

Theorem 6 *Let* $u \in \mathcal{U}_{loc}(\xi, f, g, h)$. *Assume that* $u \leq M$ *on* Γ*, then for all* $t \in [0, T]$ *:*

$$E \left\| (u - M)^+ \right\|^p_{\infty,\infty;t} \leq k(t)\, E \left(\|(\xi - M)\|^p_\infty + \left\| (f^M - b)^+ \right\|^{*p}_{\theta,t} + \left\| |g^M|^2 \right\|^{*p/2}_{\theta;t} + \left\| |(h^M - \sigma)|^2 \right\|^{*p/2}_{\theta;t} \right)$$

where $f^M(t,x) = f(t,x,M_t,0)$, $g^M(t,x) = g(t,x,M_t,0)$, $h^M(t,x) = h(t,x,M_t,0)$.

Remark 1 *If* M *is a constant, then using the Lipschitz condition on the coefficients, we get an estimate similar to the classical one for quasilinear PDE obtained bay Arronson and Serrin .*

Step 1 : Estimates for solutions with null Dirichlet conditions under weaker L^1-integrability condition on f^0

Theorem 7 *There exists a unique solution of (1) in $\mathcal{H}_0$. Moreover, this solution has a version with $L^2(\mathcal{O})$-continuous trajectories and it satisfies the following estimates*

$$E\left(\|u\|_{2,\infty;t}^2+\|\nabla u\|_{2,2;t}^2\right)\leq k\left(t\right)E\left(\|\xi\|_2^2+\left(\left\|f^0\right\|_{\#;t}^*\right)^2+\left\|g^0\right\|_{2,2;t}^2+\left\|h^0\right\|_{2,2;t}^2\right),$$

for each $t\geq 0$, where $k\left(t\right)$ is a constant that only depends on the structure constants and t.

Proof : We start by writing Ito's formula for the solution in the form

$$\begin{aligned}&\|u_t\|_2^2+2\int_0^t\mathcal{E}\left(u_s,u_s\right)ds=\|\xi\|_2^2+2\int_0^t\left(u_s,f_s\left(u_s,\nabla u_s\right)\right)ds\\&-2\int_0^t\sum_{i=1}^d\left(\partial_iu_s,g_{i,s}\left(u_s,\nabla u_s\right)\right)ds+\int_0^t\left\|h_s\left(u_s,\nabla u_s\right)\right\|_2^2ds\\&+2\sum_{j=1}^{d_1}\int_0^t\left(u_s,h_{j,s}\left(u_s,\nabla u_s\right)\right)dB_s^j,\end{aligned}\tag{4}$$

equality which holds a.s. The Lipschitz condition and the inequality (2) lead to the following estimate

$$\int_0^t (u_s, f_s(u_s, \nabla u_s))\, ds \le \varepsilon \|\nabla u\|_{2,2;t}^2 + c_\varepsilon \|u\|_{2,2;t}^2 + \delta \|u\|_{\#;t}^2 + c_\delta \left(\left\| f^0 \right\|_{\#;t}^* \right)^2,$$

where $\varepsilon, \delta > 0$ are two small parameters to be chosen later and c_ε, c_δ are constants depending of them. Similar estimates hold for the next two terms

$$-\int_0^t \sum_{i=1}^d (\partial_i u_s, g_{i,s}(u_s, \nabla u_s))\, ds \le (\alpha + \varepsilon) \|\nabla u\|_{2,2;t}^2 + c_\varepsilon \|u\|_{2,2;t}^2 + c_\varepsilon \left\| g^0 \right\|_{2,2;t}^2,$$

$$\int_0^t \|h_s(u_s, \nabla u_s)\|_2^2\, ds \le \left(\beta^2 + \varepsilon\right) \|\nabla u\|_{2,2;t}^2 + c_\varepsilon \|u\|_{2,2;t}^2 + c_\varepsilon \left\| h^0 \right\|_{2,2;t}^2.$$

Corollary 8 *Let us assume the hypotheses of the preceding Theorem with the same notations. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ and assume that φ'' is bounded and $\varphi'(0) = 0$. Then the following relation holds a.s. for all $t \ge 0$:*

$$\int_{\mathcal{O}} \varphi(u_t(x))\, dx + \int_0^t \mathcal{E}(\varphi'(u_s), u_s)\, ds = \int_{\mathcal{O}} \varphi(\xi(x))\, dx + \int_0^t (\varphi'(u_s), f_s(u_s, \nabla u_s)\, ds$$

$$
-\int_0^t \sum_{i=1}^{d} \left(\partial_i \left(\varphi'\left(u_s\right)\right), g_{i,s}(u_s, \nabla u_s)\, ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u_s\right), \left|h_s(u_s, \nabla u_s)\right|^2\right) ds\right.
$$

$$
+\sum_{j=1}^{d_1} \int_0^t \left(\varphi'\left(u_s\right), h_{j,s}(u_s, \nabla u_s)\right) dB_s^j.
$$

Step 2 : Estimates of the positive part of the solution

We next prove an estimate for the positive part u^+ of the solution $u = \mathcal{U}(\xi, f, g, h)$. For this we need the following notation :

$$f^{u,0} = 1_{\{u>0\}} f^0, \ g^{u,0} = 1_{\{u>0\}} g^0, \ h^{u,0} = 1_{\{u>0\}} h^0,$$

$$f^u = f - f^0 + f^{u,0}, \ g^u = g - g^0 + g^{u,0}, \ h^u = h - h^0 + h^{u,0}$$

$$f^{u,0+} = 1_{\{u>0\}} \left(f^0 \vee 0\right), \ \xi^+ = \xi \vee 0.$$

Theorem 9 *The positive part of the solution satisfies the following estimate*

$$E\left(\left\|u^+\right\|_{2,\infty;t}^2 + \left\|\nabla u^+\right\|_{2,2;t}^2\right) \leq k(t) E\left(\left\|\xi^+\right\|_2^2 + \left(\left\|f^{u,0+}\right\|_{\#;t}^*\right)^2 + \left\|g^{u,0}\right\|_{2,2;t}^2 + \left\|h^{u,0}\right\|_{2,2;t}^2\right).$$

Proof : The idea is to apply Ito's formula to the function ψ defined by $\psi(y) = (y^+)^2$, for any $y \in \mathbb{R}$.

• Since this function is not of the class $\mathcal{C}^2$ we shall make an approximation as follows. Let φ be a $\mathcal{C}^\infty$ function such that $\varphi(y) = 0$ for any $y \in]-\infty, 1]$ and $\varphi(y) = 1$ for any $y \in [2, \infty[$. We set $\psi_n(y) = y^2 \varphi(ny)$, for each $y \in \mathbb{R}$ and all $n \in \mathbb{N}^*$. It is easy to verify that $(\psi_n)_{n\in\mathbb{N}^*}$ converges uniformly to the function ψ and that

$$\lim_{n\to\infty} \psi_n'(y) = 2y^+, \ \lim_{n\infty} \psi_n''(y) = 2 \cdot 1_{\{y>0\}},$$

for any $y \in \mathbb{R}$. Moreover we have the estimates

$$0 \leq \psi_n(y) \leq \psi(y), \ 0 \leq \psi'(y) \leq Cy, \ |\psi_n''(y)| \leq C,$$

for any $y \geq 0$ and all $n \in \mathbb{N}^*$, where C is a constant.

Step 3 : Extension of the Ito's formula

The following theorem represents a key technical result which leads to a generalization of the estimates of the positive part. Let $u \in \mathcal{U}_{loc}(\xi, f, g, h)$ be a solution and u^+ its positive part and set

$$f^{u,0} = 1_{\{u>0\}} f^0, \ g^{u,0} = 1_{\{u>0\}} g^0, \ h^{u,0} = 1_{\{u>0\}} h^0,$$
$$f^{u,0+} = 1_{\{u>0\}} \left(f^0 \vee 0\right), \ \xi^+ = \xi \vee 0.$$

Theorem 10 *Assume that u^+ belongs to $\mathcal{H}$ and assume that the data satisfy the following integrability conditions*

$$E\left\|\xi^+\right\|_2^2 < \infty,\ E\left(\left\|f^{u,0}\right\|_{\#;t}^*\right)^2 < \infty,\ E\left\|g^{u,0}\right\|_{2,2;t}^2 < \infty,\ E\left\|h^{u,0}\right\|_{2,2;t}^2 < \infty,$$

for each $t \geq 0$. Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a function of class $\mathcal{C}^2$, which admits a bounded second order derivative and such that $\varphi'(0) = 0$. Then the following relation holds, a.s., for each $t \geq 0$,

$$\int_{\mathcal{O}} \varphi\left(u_t^+(x)\right) dx + \int_0^t \mathcal{E}\left(\varphi'\left(u_s^+\right), u_s^+\right) ds = \int_{\mathcal{O}} \varphi\left(\xi^+(x)\right) dx + \int_0^t \left(\varphi'\left(u_s^+\right), f_s\left(u_s^+, \nabla u_s^+\right)\right) ds$$

$$-\int_0^t \sum_{i=1}^d \left(\varphi''\left(u_s^+\right) \partial_i u_s^+, g_{i,s}\left(u_s^+, \nabla u_s^+\right)\right) ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u_s^+\right), \left|h_s\left(u_s^+, \nabla u_s^+\right)\right|^2\right) ds$$

$$+\sum_{j=1}^{d_1} \int_0^t \left(\varphi'\left(u_s^+\right), h_{j,s}\left(u_s^+, \nabla u_s^+\right)\right) dB_s^j.$$

Step 4 : Estimates

Corollary 11 *Under the hypotheses of the above theorem with same notations, one has the following estimates*

$$E\left(\left\|u^{+}\right\|_{2,\infty;t}^{2}+\left\|\nabla u^{+}\right\|_{2,2;t}^{2}\right)\leq k\left(t\right)E\left(\left\|\xi^{+}\right\|_{2}^{2}+\left(\left\|f^{u,0+}\right\|_{\#;t}^{*}\right)^{2}+\left\|g^{u,0}\right\|_{2,2;t}^{2}+\left\|h^{u,0}\right\|_{2,2;t}^{2}\right).$$

As a Consequence : More general comparison Theorem

Theorem 12 *Assume that $f^1, f^{\,2}$ are two functions similar to f which satisfy the Lipschitz condition and such that both triples (f^1, g, h) and (f^2, g, h) satisfy our assumptions.*
Assume that ξ^1, ξ^2 are random variables similar to ξ.
Let $u^i \in \mathcal{U}_{loc}\left(\xi^i, f^i, g, h\right), i = 1, 2$ and suppose that the process $\left(u^1 - u^2\right)^+$ belongs to $\mathcal{H}_0$ and that one has

$$E\left(\left\|f^{1}\left(u^{2},\nabla u^{2}\right)-f^{2}\left(u^{2},\nabla u^{2}\right)\right\|_{\#;t}^{*}\right)^{2}<\infty,$$

for each $t \geq 0$.
If $\xi^1 \leq \xi^2$ a.s. and $f^1\left(u^2, \nabla u^2\right) \leq f^2\left(u^2, \nabla u^2\right)$ a.s., then one has $u^1 \leq u^2$ a.s.

Idea of the proof : The difference $v = u^1 - u^2$ belongs to $\mathcal{U}_{loc}\left(\overline{\xi}, \overline{f}, \overline{g}, \overline{h}\right)$, where $\overline{\xi} = \xi^1 - \xi^2$,

$$\overline{f}(t,\omega,x,y,z) = f^1\left(t,\omega,x,y+u_t^2(x), z+\nabla u_t^2(x)\right) - f^2\left(t,\omega,x,u_t^2(x), \nabla u_t^2(x)\right),$$

$$\overline{g}(t,\omega,x,y,z) = g\left(t,\omega,x,y+u_t^2(x), z+\nabla u_t^2(x)\right) - g\left(t,\omega,x,u_t^2(x), \nabla u_t^2(x)\right),$$

$$\overline{h}(t,\omega,x,y,z) = h\left(t,\omega,x,y+u_t^2(x), z+\nabla u_t^2(x)\right) - h\left(t,\omega,x,u_t^2(x), \nabla u_t^2(x)\right).$$

The result follows from the preceding corollary, since $\overline{\xi} \leq 0$ and $\overline{f}^0 \leq 0$ and $\overline{g}^0 = \overline{h}^0 = 0$. $\square$

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