Stochastic Control Problems for Systems Driven by Normal Martingales

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Objective:

Study of a class of control problems with dynamics:

$$Y_s = y + \int_t^s b(Y_r, u_r, \pi_r)dr + \int_t^s \sigma(Y_{r-}, u_r, \pi_r)dX_r^u,\ t \le s, \tag{1}$$

where X^u is a martingale satisfying a multidimensional Structure Equation:

$$[X^u]_t = t + \int_0^t u_s dX_s^u,\ t \ge 0. \tag{2}$$

Example-motivation:

In an optimal reinsurance and investment selection problem, the dynamics of the risk reserve of the insurance company can be described by an SDE of the following form (see, e.g., Y. Liu, J. Ma)

$$\begin{aligned} Y_t = y &+ \int_0^t b(Y_s, a_s(\cdot), \pi_s)ds \\ &+ \int_0^t \sigma(\pi_s)dW_s - \int_0^t \int_{\mathbb{R}_+} a_s(x)f(s,x)\tilde{N}(dx,ds), \end{aligned}$$

with

- W **standard** d**-dimensional Brownian motion: uncertainty of the underlying security market;**
- $\int_0^t \int_{\mathbb{R}_+} f(s,x)\tilde{N}(dx,ds)$, **with** $\tilde{N}$ **compensated Poisson random measure: accumulated incoming claims up to time** t;
- a **a random field: reinsurance policy (or "*retention ratio*");**
- $\pi = (\pi^1, \cdots, \pi^n)$: **investment portfolio.**

Some references

On financial market models:

- **Dritschel, M. and Protter, P.** *Complete markets with discontinuous security price*, **(1998) preprint.**
- **Ma, J., Protter, P., and San Martin, J.,** *Anticipating integrals for a class of martingales*, *Bernoulli* **4:1 (1998).**

On Structure Equations:

- **Emery, M.,** *On the Azéma Martingales*, **Séminaire de Probabilités XXIII, Springer Verlag, Lecture Notes in Mathematics 1372 (1989).**
- **Meyer, P.A.,** *Construction de solutions d'équations de structure*, **Séminaire de Probabilités XXIII, Springer Verlag, Lecture Notes in Mathematics 1372 (1989).**
- **Attal, S. and Emery, M.,** *Equations de structure pour les martingales vectorielles*, **Séminaire de Probabilités XXVIII, Springer Verlag, Lecture Notes in Mathematics 1583, 256-278M. (1996),** *Martingales d'Azéma bidimensionnelles. Hommage à P.-A. Meyer et J. Neveu*, **Astérisque 236 (1994)**
- **Taviot, G.,** *Martingales et équations de structure : étude géométrique* **Thèse, IRMA, Strasbourg, 1999.**
- **Kurtz, D.,** *Une caractérisation des martingales d'Azéma bidimensionnelles de type II*, **Séminaire de Probabilités XXXV, Springer Verlag, Lecture Notes in Mathematics 1755 (2001).**

Outline

1. *Introduction*
2. ► *Preliminaries*
    - *Normal Martingales and Structure Equations*
    - *The Wiener-Poisson space*
3. *The Stochastic Control Problem*
4. *The HJB Equation*

Normal Martingales and Structure Equations.

Let $(\Omega, \mathcal{F}, P, \mathbf{F})$ **a filtered probability space, on which is defined** $X = (X_t)_{t\geq 0}$ **a real, square integrable martingale.**
Quadratic variation : $t \geq 0$**,**
$\Delta : 0 = t_0 < \ldots < t_n = t, \ |\Delta| = \sup_i |t_{i+1} - t_i|$**.**

$$[X]_t = \lim_{|\Delta|\to 0} \sum_{i=0}^{n-1} (X_{t+1} - X_{t_i})^2,$$
$$\langle X \rangle_t = \lim_{|\Delta|\to 0} \sum_{i=0}^{n-1} E[(X_{t+1} - X_{t_i})^2 | \mathcal{F}_{t_i}].$$

Definition: X **is normal if** $\langle X \rangle_t = t$**, for all** $t \geq 0, P$**-a.s. .**

We have also **:** $[X]_t = \langle X^c \rangle_t + \sum_{0\leq s\leq t} \Delta X_s^2$**.**

Definition: (Emery 89) A normal martingale X **satisfies a structure equation if there exists** u $\mathbf{F}$**-predictable, such that**

$$[X]_t - t = \int_0^t u_s dX_s,$$

or equivalently

$$d[X]_t = dt + u_t dX_t.$$

Normal Martingales and Structure Equations.

Structure equation : $d[X]_t = dt + u_t dX_t \quad (1)$.

***Examples* :**

- $u \equiv 0$ **:** X **is a Brownian motion,**
- $u \equiv \alpha \in \mathbb{R}^*$ **:** $X_t = \alpha(N_{t/\alpha^2} - t/\alpha^2)$**, where** N **is a standard Poisson process,**
- $u_t = -X_{t-}$ **:** X **is an Azéma martingale.**

Main Properties: Let X **be a solution of (1).**
Then,

- **if** $\Delta X_t \neq 0$**, then** $\Delta X_t = u_t$**;**
- **continuous part :** $dX_t^c = \mathbf{1}_{\{u_t=0\}} dX_t$**;**
- **pure jump part :** $dX_t^d = \mathbf{1}_{\{u_t\neq 0\}} dX_t$**.**

The Wiener-Poisson space.

We consider now, for $T > 0$,

- $B = (B_t)_{t \geq 0}$, **a d-dimensional Brownian motion,**
- μ, **a Poisson random measure on $[0, T] \times I\!R^*$, with Lévy measure ν:**

$$\int_{I\!R^*} (1 \wedge x^2) \nu(dx) < +\infty,$$

with B and μ independent.
Let $\mathbf{F}$ be the augmented version of the filtration generated by B and μ.

Martingale representation property :
$\forall X \in \mathcal{M}^2(\mathbf{F}, P, I\!R^d)$ **with $X_0 = 0$,**
$\exists !\ \alpha \in L^2([0, T]; I\!R^{d \times d}), \beta \in L^2([0, T] \times I\!R^*; dtd\nu; I\!R^d)$ **s.t.**

$$X_t = \int_0^t \alpha_s dB_s + \int_0^t \int_{I\!R^*} \beta_s(x) \tilde{\mu}(dsdx), \quad t \in [0, T],$$

with $\tilde{\mu}(dtdx) = \mu(dtdx) - \nu(dx)dt, \ (t, x) \in [0, T] \times I\!R^*$.

The Wiener-Poisson space.

Proposition: **Let**

- $(u_t)_{t\geq 0}$, **a bounded, F-predictable process taking values in** $I\!R^d$,
- X, **a** d**-dimensional martingale, with**

$$X_t = \int_0^t \alpha_s dB_s + \int_0^t \int_{I\!R^*} \beta_s(x)\tilde{\mu}(dsdx), \quad t \in [0,T].$$

Then, X **satisfies a structure equation**

$$\begin{cases} d[X^i]_t = dt + u_t^i dX_t^i, & 1 \leq i \leq d, \\ d[X^i, X^j]_t = 0, & 1 \leq i < j \leq d, \ t \geq 0, \end{cases}$$

iff there are random sets $A_s^i \in \mathcal{B}(I\!R^*) \otimes \mathcal{F}_s$, $s \geq 0$, $1 \leq i \leq d$, **such that**

- **(i)** $\sum_{k=1}^d \alpha_s^{i,k}\alpha_s^{j,k} = \delta_{i,j}\mathbf{1}_{\{u_t^i=0\}}$, $dt \times dP$**-a.e.;**
- **(ii)** $\beta_t^i(x) = u_t^i \mathbf{1}_{A_t^i}(x)$, $dt \times d\nu \times dP$**-a.e.;**
- **(iii)** $\nu(A_t^i \cap A_t^j)\mathbf{1}_{\{u_t^i\neq 0, u_t^j \neq 0\}} = \delta_{i,j}\,\frac{1}{(u_t^i)^2}\mathbf{1}_{\{u_t^j\neq 0\}}$, $dt \times dP$**-a.e.,** $1 \leq i,j \leq d$**.**

Corollary: **Assume:** $\nu^c([-1,1]\setminus\{0\})=\infty$.
Then, $\forall u$ **bounded, F-predictable,** $I\!R^d$**-valued, the structure equation**

$$\begin{cases} d[X^i]_t = dt + u^i_t dX^i_t, & 1\le i\le d, \\ d[X^i,X^j]_t = 0, & 1\le i<j\le d,\ t\ge 0, \end{cases}$$

has at least one solution in the Wiener-Poisson space.

Remark : This solution is not necessarily unique.

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The Stochastic Control Problem

Let $T > 0$ a finite time horizon and $(\Omega, \mathcal{F})$ the canonical Wiener-Poisson space :
$\Omega = \Omega_1 \times \Omega_2$**, with**
$\Omega_1 = C_0([0,T], I\!R^d)$ **and** $\Omega_2 = \{q \ \ \overline{N}$**-valued measures on** $[0,T] \times I\!R^*\}$**.**
$\mathcal{F}_s^t = \sigma\{B_{r \vee t} - B_t, \ \mu(A), \ A \in \mathcal{B}([t, r \vee t] \times I\!R^*), 0 \leq r \leq s\} \vee \mathcal{N}_P$**.**

Let ν be a Lévy measure s.t.

$$\int_{I\!R^*} (1 \wedge x^2) \nu(dx) < +\infty, \ \nu^c([-1,1] \setminus \{0\}) = +\infty,$$

and P a probability on $(\Omega, \mathcal{F})$ s.t. $B_t(\omega, q) := \omega(t)$ is a Brownian motion, $\mu(\omega) = q$ is a Poisson random measure with Lévy measure ν, and B and μ are independent.

Let U_1 **(resp.** U^d**) a non empty compact set in** $\mathbb{R}$ **(resp.** $\mathbb{R}^d$**).**
Set $\overline{U} = U_1 \times U^d$**,**
let $t \in [0, T]$

Definition: (π, u, X) **is a control at time** t **if:**

- (π, u) **is a pair of** $\mathbf{F}^t$**-predictable processes with values in** $\overline{U}$**,**
- $X \in \mathcal{M}^2(\mathcal{F}, \mathbb{R}^d)$ **satisfies**

$$\begin{cases} [X^i]_s = s + \int_0^s u_r^i dX_r^i, \ 1 \leq i \leq d, \ s \in [0, T], \\ [X^i, X^j]_s = 0, \ 1 \leq i < j \leq d, \ s \in [0, T], \\ X_s - X_t \perp\!\!\!\perp \mathcal{F}_t, \ \forall s \geq t. \end{cases}$$

Let $\mathcal{U}(t)$ **be the set of controls at time** t**.**

The Stochastic Control Problem

Let $(t,y) \in [0,T] \times I\!R^m$**,** $a = (\pi, u, X) \in \mathcal{U}(t)$**,**

let $Y^{t,y,a}$ **be the unique solution of**

$$\begin{cases} Y_s = y + \int_t^s b(Y_r, \pi_r, u_r)dr + \int_t^s \sigma(Y_{r-}, \pi_r, u_r)dX_r, \quad s \in [t,T], \\ Y_s = y, \ \ s < t. \end{cases}$$

Cost functional : $J(t,y,a) = E[g(Y_T^{t,y,a})], \quad (t,y) \in [0,T] \times I\!R^m$**,**

Value function : $V(t,y) = \inf_{a \in \mathcal{U}(t)} E[g(Y_T^{t,y,a})] \ \ (t,y) \in [0,T] \times I\!R^m$**,**

(with classical assumptions on b**,** σ**,** g**).**

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Theorem: **The value function V is solution, in the sense of viscosity, of**

$$(*) \begin{cases} -\frac{\partial}{\partial t}V(t,y) - \inf_{(\pi,u)\in\overline{U}} \mathcal{L}_{\pi,u}[V](t,y) = 0, \ (t,y) \in [0,T] \times I\!R^m, \\ V(T,y) = g(y), \ y \in I\!R^m, \end{cases}$$

with

$$\begin{aligned} \mathcal{L}_{\pi,u}[V](t,y) = \ & \nabla_y V(t,y) b(y,\pi,u) \\ & + \sum_{i=1}^{d} \{ \mathbf{1}_{\{u^i=0\}} \tfrac{1}{2} (D^2_{yy} V(t,y) \sigma^i(y,\pi,u), \sigma^i(y,\pi,u)) \ . \\ & + \mathbf{1}_{\{u^i \neq 0\}} \frac{V(t,y+u^i\sigma^i(y,\pi,u)) - V(t,y) - u^i \nabla_y V(t,y) \sigma^i(y,\pi,u)}{(u^i)^2} \}. \end{aligned}$$

Condition H : **there exists a compact set** $U \in I\!R$ **such that**

- $0 \notin U$,
- $U = U_0$ **or** $U_0 = \{0\} \cup U$.

Theorem : **Under the condition H, the value function** V **is the unique solution of equation (*) among all bounded, continuous functions.**