# Orthogonal Polynomials and Random Matrices

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# 1 Linear algebra and Analysis

## 1.1 Self-adjoint operators

Let $\mathcal{H}$ an Hilbert space, $A$ a self-adjoint bounded operator and $e$ a cyclic* unit vector. We say that

$$(\mathcal{H}, A, e) \sim (\mathcal{K}, B, f)$$

iff there exits an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $B = VAV^{-1}$ and $f = Ve$. For each equivalence class, there is a unique probability with compact support $\mu$ such that:

$$\langle e, A^n e\rangle_{\mathcal{H}} = \int x^n d\mu(x) \quad , \quad n = 1, 2, \dots$$

2 remarkable elements in a class :

1) $(L^2(d\mu),\ h \mapsto (x \mapsto xh(x))\ , \mathbf{1})$

2) $(\ell^2, J, \varepsilon_1)$ where $J$ is a tridiagonal matrix and $\varepsilon_1 = (1, 0, 0, \dots)$.

<u>Finite dimension</u> Let $A$ be a self-adjoint $N \times N$ matrix, with eigenvalues $(\lambda_j)_{j=1}^N$ and eigenvectors $(\psi_j)_{j=1}^N$.

i) Set $w_j := |\langle \psi_j, e_1 \rangle|^2$. Then

$$\mu_w^{(N)} = \sum_{j=1}^N w_j \delta_{\lambda_j} \ .$$

Besides, the ESD is

$$\mu^{(N)} = \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j}$$

ii) In the basis $(\epsilon_1, \dots, \epsilon_N)$ obtained applying Gram-Schmidt orthonormalization to $(e_0, Ae_0, \dots, A^{N-1}e_0)$, the matrix $A$ becomes

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_{N-1} & b_N \end{pmatrix}$$

with $a_j > 0$.

iii) If

$$(e_0, Ae_0, \dots, A^{N-1}e_0) \leftrightarrow (1, x, \dots, x^{N-1})$$

then, by Gram-Schmidt orthonormalization

$$(\epsilon_1, \dots, \epsilon_N) \leftrightarrow (1, p_1(x), \dots, p_{N-1}(x))$$

They satisfy the three-term recursion

$$xp_j(x) = a_{j+1}p_{j+1}(x) + b_{j+1}p_j(x) + a_jp_{j-1}(x)\,.$$

If we do not normalize polynomials, we get

$$xP_j(x) = P_{j+1}(x) + b_{k+1}P_k(x) + a_k^2 P_{k-1}(x),\ j = 1, \dots N-2$$

Moreover, if we push the recursion one step further, we get the characteristic polynomial

$$P_N(x) = \prod_{k=1}^{N} (x - \lambda_k)\,.$$

The Stieltjès transform of $\mu_w^{(N)}$ is

$$\begin{aligned} m(z) &= \int \frac{d\mu_w^{(N)}(x)}{z-x} \\ &= \cfrac{1}{b_1 - z - \cfrac{a_1^2}{b_2 - z - \cfrac{a_2^2}{\dots}}} \end{aligned}$$

## 1.2 Unitary operators

Let $\mathcal{H}$ an Hilbert space, $U$ a unitary operator and $e$ a cyclic** unit vector. We say that

$$(\mathcal{H}, U, e) \sim (\mathcal{K}, W, f)$$

iff there exits an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $B = VAV^{-1}$ and $f = Ve$. For each equivalence class, there is a unique probability $\mu$ on $\mathbb{T} = \{z : |z| = 1\} = \{e^{i\theta}, \theta \in [0, 2\pi[\}$ such that

$$\langle e, U^k e\rangle_{\mathcal{H}} = \int_{\mathbb{T}} e^{ik\theta} d\mu(\theta) \quad , \quad k = 0, 1, \ldots$$

2 remarkable elements in a class :

1) $(L^2(d\mu),\ h \mapsto (z \mapsto zh(z))\ , \mathbf{1})$

2) $(\ell^2, J, \varepsilon_1)$ where $J$ is a pentadiagonal matrix and $\varepsilon_1 = (1, 0, 0, \ldots)$.

<u>Finite dimension</u> Let $U$ be a unitary $N \times N$ matrix. If $\psi_j, j = 1, \dots, N$ is a basis of eigenvectors associated to the eigenvalues $\lambda_j = e^{i\theta_j}$ we have

$$\mu = \sum_{j=1}^{N} w_j \delta_{\lambda_j} \quad , \ w_j := |\langle \psi_j, e_1 \rangle|^2$$

Starting from $1, z, z^2, \dots, z^{N-1}$, Gram-Schmidt gives <u>monic</u> polynomials $\Phi_0 = 1, \Phi_1, \dots, \Phi_{N-1}$ . Let also the transformation :

$$\Phi_k(z) = z^k + \sum_{j=1}^{k} c_j z^{k-j} \ \mapsto \ \Phi_k^\star(z) = 1 + \sum_{j=1}^{k} \bar{c}_j z^j \, .$$

We have the Schur's recursion :

$$\begin{aligned} \Phi_{k+1}(z) &= z\Phi_k(z) - \bar{\alpha}_k \Phi_k^\star(z) \\ \Phi_{k+1}^\star(z) &= -\alpha_k z \Phi_k(z) + \Phi_k^\star(z) \ \ , \ \alpha_k = -\overline{\Phi_{k+1}(0)} \, . \end{aligned}$$

The $\alpha_k$'s : coefficients of Schur (resp. Levinson, Verblunsky).

Actually

$$\alpha_0, \dots, \alpha_{N-2} \in \mathbb{D} = \{z : |z| < 1\} \ \ , \ \alpha_{N-1} \in \mathbb{T} \, .$$

Pushing the recursion we get

$$\Phi_N(z) = \prod_{j=1}^{N} (z - \lambda_j) \, . \tag{1}$$

The system $\Phi_k$ is not always total. If we orthonormalize $1, z, z^{-1}, z^2, z^{-2}, \dots$ we get

$$\chi_k(z) = \begin{cases} -z^{k/2} \varphi_k^\star(z) & : k \text{ even} \\ z^{-(k-1)/2} \varphi_k(z) & : k \text{ odd} \end{cases}$$

In the basis $(\chi_k)$, the matrix becomes pentadiagonal :

$$
\mathcal{C} = \begin{pmatrix}
\bar{\alpha}_0 & \rho_0\bar{\alpha}_1 & \rho_0\rho_1 & 0 & 0 & 0 & \dots \\
\rho_0 & -\alpha_0\bar{\alpha}_1 & -\alpha_0\rho_1 & 0 & 0 & 0 & \dots \\
0 & \rho_1\bar{\alpha}_2 & -\alpha_1\bar{\alpha}_2 & \rho_2\bar{\alpha}_3 & \rho_2\rho_3 & 0 & \dots \\
0 & \rho_1\rho_2 & -\alpha_1\rho_2 & -\alpha_2\bar{\alpha}_3 & -\alpha_2\rho_3 & 0 & \dots \\
0 & 0 & 0 & \rho_3\bar{\alpha}_4 & -\alpha_3\bar{\alpha}_4 & \rho_4\bar{\alpha}_5 & \dots \\
0 & 0 & 0 & \rho_3\rho_4 & -\alpha_3\rho_4 & \alpha_4\bar{\alpha}_5 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix}
$$

with $\rho_k = \sqrt{1 - |\alpha_k|^2}$.

# 2 Random Matrix Ensembles

## 2.1 GOE

GOE = [Symmetric matrices with independent $\mathcal{N}(0,1)$ coefficients]

- The eigenvalues have a joint density proportional to

$$\left|\Delta(\lambda_1, \ldots, \lambda_N)\right| e^{-\frac{1}{2}\sum_{i=1}^N \lambda_i^2}$$

with $\Delta$ Vandermonde.

- The two $N$-uples $(\lambda_1, \ldots, \lambda_N)$ and $(w_1, \ldots, w_N)$ are independent,
- The law of $(w_1, \ldots, w_N)$ is $\mathrm{Dir}(1/2)$ on the simplex.

**Theorem 2.1 (Dumitriu Edelman 02)** *The Jacobi coefficients $a_1, \ldots, a_{N-1}, b_1, \ldots, b_N$ are independent, the $b_k$'s are $\mathcal{N}(0,1)$, and $a_k^2 \stackrel{(d)}{=} Gamma\left(\frac{N-k}{2}\right)$, $k = 1, \ldots, N-1$.*

## 2.2 CUE

We equip $\mathbb{U}(N)$ with the Haar distribution. Then

- The two $N$-uples $(\lambda_1, \dots, \lambda_N)$ and $(w_1, \dots, w_N)$ are independent.
- The $\lambda_j = e^{i\theta_j}$ have a joint density proportional to

$$Z_{n,2}^{-1} \, |\Delta(e^{i\theta_1}, \dots, e^{i\theta_N})|^2 \tag{2}$$

- The vector $(a_1, \dots, a_N)$ is uniform on the simplex.

**Theorem 2.2 (Killip Nenciu 04)** *The vector $e_1$ is a.s. cyclical and the Verblunsky coefficients of the random spectral measure of the pair $(U, e_1)$ are independent,*

$$\begin{aligned} \alpha_k &\overset{(d)}{=} e^{i\eta_k} \sqrt{Beta\Big(1, (N-k-1)\Big)} \ , \ 0 \le k \le N-2, \\ \alpha_{N-1} &\overset{(d)}{=} e^{i\eta_{N-1}} \ , \end{aligned}$$

*where the $\eta_k$'s are uniform on $[0, 2\pi]$.*

Characteristic polynomial

- Keating Snaith (CMP 00)
- Bourgade, Hughes, Nikeghbali, Rouault, Yor (arXiv 07)

**Theorem 2.3** *If $U \in \mathbb{U}(N)$ is sampled with the Haar measure, then for every $\theta \in [0, 2\pi]$*

$$\Phi_N(e^{i\theta}) = \det(e^{i\theta} I - U) = \prod_{k=0}^{N-1} (e^{i\theta} - y_k(e^{i\theta})) \tag{3}$$

*where the r.v. $y_k(e^{i\theta}), j = 0, \ldots, N-1$ are independent,*

$$\begin{aligned} y_k(\theta) &\overset{(d)}{=} e^{i\eta_k} \sqrt{Beta(1, N-k-1)}\ ,\ 0 \leq k \leq N-2, \\ y_{N-1}(\theta) &\overset{(d)}{=} e^{i\eta_{N-1}}\ , \end{aligned}$$

*where the $\eta_k$ is uniform on $[0, 2\pi]$.*

**Lemma 2.4** *We have*

$$\Phi_N(z) = \prod_{k=0}^{N} (z - y_k(z)) \,.$$

*where* $y_0 = \bar{\alpha}_0$ , $\widetilde{y}_0 = \alpha_0$ *and*

$$\begin{aligned} y_k(z) &= \bar{\alpha}_k \prod_{j=0}^{k-1} \frac{1 - z\widetilde{y}_j(z)}{z - y_j(z)} \\ \widetilde{y}_k(z) &= \alpha_k \prod_{j=0}^{k-1} \frac{z - y_j(z)}{1 - z\widetilde{y}_j(z)} \end{aligned}$$

*If* $|z| = 1$ *then*

$$|y_k(z)| = |\widetilde{y}_k(z)| = |\alpha_k| \,. \tag{4}$$

## 2.3 Extensions

Orthogonal, unitary, symplectic : $\beta = 1, 2, 4$.

What about $\beta > 0$?

### 2.3.1 G$\beta$E ensembles

Coulomb gas on $\mathbb{R}$: $N$ points on $\mathbb{R}$, with joint density

$$Z_{n,\beta}^{-1} \, |\Delta(x_1, \dots, x_N))|^{\beta} \; e^{-\frac{1}{2}\sum_{k=1}^{N} x_k^2}$$

Pb: matrix realization.

Dumitriu and Edelman found tridiagonal matrix with properties as above, with $a_k^2 \overset{(d)}{=} Gamma\left(\frac{\beta(n-k)}{2}\right)$. Moreover, the vector of weights $(w_1, \dots, w_N)$ has a Dirichlet- $\beta/2$ distribution on the simplex.

### 2.3.2 C$\beta$E ensembles

Coulomb gas on $\mathbb{T}$: $N$ points on $\mathbb{T}$, with joint density

$$Z_{n,\beta}^{-1} \, |\Delta(e^{i\theta_1}, \ldots, e^{i\theta_N})|^\beta$$

Same problem.

$$\begin{aligned} \alpha_k &\overset{(d)}{=} e^{i\eta_k} \sqrt{\mathrm{Beta}\Big(1, \frac{\beta}{2}(N-k-1)\Big)} \ , \ 0 \le k \le N-2, \\ \alpha_{N-1} &\overset{(d)}{=} e^{i\eta_{N-1}} \ \ (\eta_k \text{ uniform}) \, . \end{aligned}$$

Again, the vector of weights $(w_1, \ldots, w_N)$ has a Dirichlet- $\beta/2$ distribution on the simplex.

### 2.3.3 J$\beta$E ensembles

Coulomb gas on $\mathbb{R}$: $N$ points on $\mathbb{R}$, with joint density

$$Z_{n,\beta}^{-1} \, |\Delta(x_1, \dots, x_N)|^\beta \prod_{k=1}^{N} (1 - x_k)^{u - \frac{1}{2}} (1 + x_k)^{v - \frac{1}{2}}$$

Pb: matrix realization.

**Proposition 2.5 (K-N)** *Let the independent r.v.*

$$\begin{aligned} \alpha_{2k} &\overset{(d)}{=} Beta_s\Big(\frac{\beta}{2}(N-k-1)+u+1, \frac{\beta}{2}(N-k-1)+v+1\Big) \\ \alpha_{2k-1} &\overset{(d)}{=} Beta_s\Big(\frac{\beta}{2}(N-k-1)+u+v+2, \frac{\beta}{2}(N-k)\Big) \end{aligned}$$

*pour* $k \leq N-1$ *et* $\alpha_{2N-1} = \alpha_{-1} = -1$. *Set*

$$\begin{aligned} b_{k+1} &= (1-\alpha_{2k-1})\alpha_{2k} - (1+\alpha_{2k-1})\alpha_{2k-2} \\ a_{k+1} &= \{(1-\alpha_{2k-1})(1-\alpha_{2k}^2)(1+\alpha_{2k+1})\}^{1/2} \end{aligned}$$

*Then the eigenvalues of the tridiagonal matrix* $(a_k, b_k)$ *are* $J\beta E(u,v,N)$ *distributed.*

# 3 Asymptotic results

## 3.1 Characteristic polynomials

**Theorem 3.1 (Bourgade, Nikeghbali, Rouault 07)** *1. As $N \to \infty$*

$$\left(\log \Phi_{\lfloor Nt \rfloor}(1);\ t \in [0,1)\right) \Rightarrow \left(\mathbf{B}_{-\frac{1}{2}\log(1-t)};\ t \in [0,1)\right), \tag{5}$$

*where* $\mathbf{B}$ *is a standard complex Brownian motion and* $\Rightarrow$ *stands for the weak convergence of distributions in the set of càdlàg functions on* $[0,1)$*, starting from* $0$*.*

*2. As $N \to \infty$,*

$$\frac{\log \Phi_N(1)}{\sqrt{2 \log N}} \Rightarrow \mathcal{N}_1 + i\mathcal{N}_2 \tag{6}$$

*where* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are independent standard normal and independent of* $\mathbf{B}$*.*

## 3.2 Spectral measures

### 3.2.1 Classical results on $\mu^{(N)}$

Let

$$\Sigma(\xi) = -\int\int \log(x-y)d\xi(x)d\xi(y)$$

**Theorem 3.2** *1. (Ben Arous-Guionnet 97) If* $A^{(N)} \overset{(\mathrm{d})}{=} \frac{1}{\sqrt{\beta N}} G\beta E(N)$*, the sequence* $(\mu^{(N)})$ *satisfies a LDP* *with speed* $N^2$ *and good rate function* $(\xi \in \mathcal{M}_1(\mathbb{R}))$

$$I(\xi) = \Sigma(\xi) - C_1 \int x^2 d\xi(x) + C_2 \,.$$

*2. (Hiai-Petz 00) If* $U^{(N)} \overset{(\mathrm{d})}{=} C\beta E^{(N)}$*, the sequence* $(\mu^{(N)})$ *satisfies a LDP* *with speed* $N^2$ *and good rate function* $(\xi \in \mathcal{M}_1(\mathbb{T}))$

$$I(\xi) = \Sigma(\xi) \,.$$

*3. (Hiai-Petz 06) If* $A^{(N)} \stackrel{(\mathrm{d})}{=} J\beta E(\tau_1 N, \tau_2 N, N)$*, the sequence* $(\mu^{(N)})$ *satisfies a LDP* *with speed* $N^2$ *and good rate function* $(\xi \in \mathcal{M}_1([-1,1]))$

$$I(\xi) = C_1 \Sigma(\xi) + \int [C_2 \log(1-x) + C_3 \log x]\, d\xi(x) + C_4\,.$$

The equilibrium measures are

C$\beta$E : UNIF is the uniform distribution on $\mathbb{T}$ in the C$\beta$E

G$\beta$E : SEMICIRCLE is the distribution on $[-2,2]$ with density $x \mapsto \frac{1}{2\pi}\sqrt{4-x^2}$

J$\beta$E : McKAY is the Kesten-McKay distribution with density

$$C_{u_-,u_+} \frac{\sqrt{(x-u_-)(u_+-x)}}{2\pi(1-x^2)} 1\!\!1_{(u_-,u_+)}(x) \tag{7}$$

where $u_\pm$ depend on $\tau_1, \tau_2$.

### 3.2.2 (New) results on $\mu_w^{(N)}$

[Gamboa, Lozada, Rouault 07] Let $\mathcal{K}$ be the Kullback entropy :

$$\begin{cases} \mathcal{K}(P,Q) & = \int_U \log \frac{dP}{dQ} dP \quad \text{if } P \ll Q \ \text{ and } \ \log \frac{dP}{dQ} \in L^1(P), \\ & = +\infty \quad \text{otherwise.} \end{cases} \tag{8}$$

**Theorem 3.3** *1. If $U^{(N)} \stackrel{(\mathrm{d})}{=} C\beta E^{(N)}$, the sequence $(\mu_w^{(N)})$ satisfies a LDP on $\mathcal{M}_1(\mathbb{T})$ with speed $N$ and good rate function*

$$I(\xi) = \frac{\beta}{2} \mathcal{K}\left(UNIF, \xi\right) . \tag{9}$$

*2. If $A^{(N)} \stackrel{(\mathrm{d})}{=} \frac{1}{\sqrt{\beta N}} G\beta E(N)$, the sequence $(\mu_w^{(N)})$ satisfies a LDP on $\mathcal{M}_1(\mathbb{R})$ with speed $N$ and good rate function*

$$I(\xi) = \frac{\beta}{2} \mathcal{K}\left(SEMICIRCLE, \xi\right) . \tag{10}$$

*3. If* $A^{(N)} \stackrel{(\mathrm{d})}{=} J\beta E(\tau_1 N, \tau_2 N, N)$, *the sequence* $(\mu_w^{(N)})$ *satisfies a LDP on* $\mathcal{M}_1([-1,1])$ *with speed* $N$ *and good rate function*

$$I(\xi) = \frac{\beta}{2}\mathcal{K}\left(McKAY, \xi\right) . \tag{11}$$