On the Kertèsz line: Thermodynamic versus Geometric phase transitions

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works with

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Summary

- What is the Kertèsz ($2D$–Ising model)
- Some Results for Kertèsz line for Potts model (analytical and numerical)
    - Potts model and some known results
    - Various representations of Potts models
    - Results
    - Scheme of proofs
- "Kertèsz line" for Potts model on the complete graph
    - Erdös–Renyi random graph
    - Known results for Curie–Weiss Potts model
    - Various representations
    - Known results for Fortuin–Kasteleyn representation
    - What's we expect

The Kertèsz Line

(Kertèsz 1989, Stauffer and Aharony, Percolation Theory: the trouble with Kertèsz)

$2-D$ Ising model
$\sigma_i = \pm 1, \qquad i \in \Lambda \subset \mathbb{Z}^2$

+	−	+	+	+	−
+	+	−	−	+	−
−	−	+	+	+	−
+	+	−	−	+	+
+	−	−	+	−	+

Boltzmann weight

$$\begin{aligned}\omega_{\text{Ising}}(\boldsymbol{\sigma}) &= \prod_{\langle i,j\rangle} e^{\frac{\beta}{2}(\sigma_i\sigma_j - 1)} \prod_i e^{\frac{h}{2}(\sigma_i - 1)} \\ &= \prod_{\langle i,j\rangle} e^{\beta(\delta_{\sigma_i,\sigma_j} - 1)} \prod_i e^{h(\delta_{\sigma_i,1} - 1)}\end{aligned}$$

$\beta = \frac{1}{kT}$, inverse temperature
h external magnetic field
product is over nearest neighbour pairs.

- $h = 0$. Phase transition

$$(e^{\beta_c} - 1)^2 = 2, \qquad \beta_c = \ln(1 + \sqrt{2})$$

  - $\beta > \beta_c$

→ positive spontaneous magnetization

→ Two translation invariant Gibbs states

→ Positive surface tension between the two states

  - $\beta < \beta_c$

→ no spontaneous magnetization

→ Unique phase

- $h > 0$ "Nothing happens"

  - For all β

→ Analytic free energy

→ Unique phase

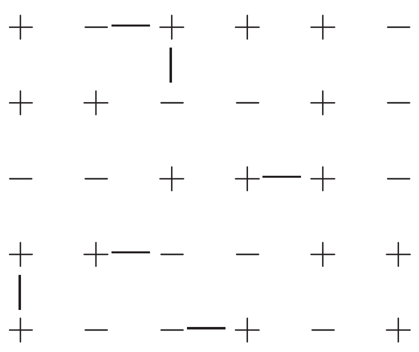
Edwards–Sokal representation

$$\omega_{\mathrm{ES}}(\boldsymbol{\sigma}, \boldsymbol{\eta}) = \prod_{\langle i,j \rangle} [e^{-\beta} \delta_{\eta_{ij},0} + (1 - e^{-\beta}) \delta_{\eta_{ij},1} \delta_{\sigma_i,\sigma_j}] \times \prod_i e^{h(\delta_{\sigma_i,1} - 1)}$$

where the RV $\eta_{ij} \in \{0, 1\}$

: write for each edge $\langle i, j \rangle$:

$$\begin{aligned} e^{\beta(\delta_{\sigma_i,\sigma_j} - 1)} &= e^{-\beta} + (1 - e^{-\beta}) \delta_{\sigma_i,\sigma_j} \\ &= \sum_{\eta_{ij}=0,1} e^{-\beta} \delta_{\eta_{ij},0} + (1 - e^{-\beta}) \delta_{\sigma_i,\sigma_j} \delta_{\eta_{ij},1} \end{aligned}$$

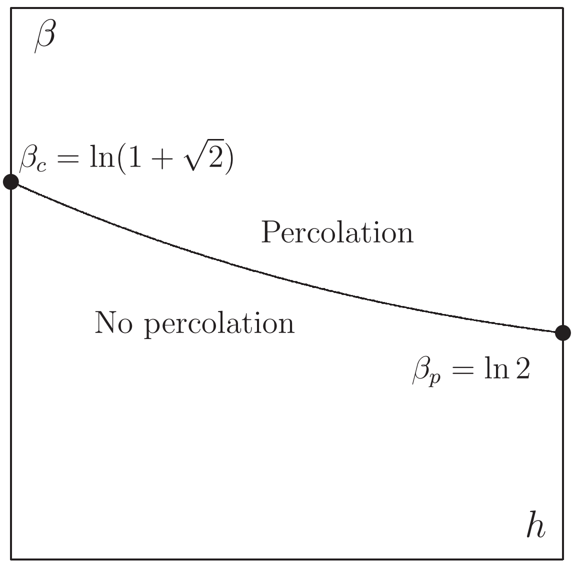


- $h = \infty$ all $\sigma_i = +1$

$$\omega_{\mathrm{ES}}(\boldsymbol{\eta}) = \prod_{\langle i,j \rangle} [e^{-\beta} \delta_{\eta_{ij},0} + (1 - e^{-\beta}) \delta_{\eta_{ij},1}]$$

→ Usual $2 - D$ bond Percolation problem with parameter $e^{-\beta}$

→ Percolation transition at $e^{-\beta_p} = 1/2$, $\beta_p = \ln 2$


β
$\beta_c = \ln(1+\sqrt{2})$
Percolation
No percolation
$\beta_p = \ln 2$
h

Potts model

$\sigma_i \in \{1, ..., q\}$ $\qquad$ $i \in \Lambda \subset \mathbb{Z}^d$

1	1	3	6	7	2
4	1	1	4	4	3
2	6	3	3	2	1
3	4	2	6	5	4
3	4	4	1	2	2

Boltzmann weight

$$\omega_{\text{Potts}}(\boldsymbol{\sigma}) = \prod_{\langle i,j \rangle} e^{\beta(\delta_{\sigma_i,\sigma_j}-1)} \prod_i e^{h(\delta_{\sigma_i,1}-1)}$$

- $h = 0$. q large First order phase transition

$d=2$: $(e^{\beta_c} - 1)^2 = q$, $\beta_c = \ln(1+\sqrt{q})$

$d \geqslant 3$: $\beta_c \simeq \frac{1}{d} \ln q$

– $\beta > \beta_c$

→ positive spontaneous magnetization

→ q ordered translation invariant Gibbs states

→ Positive surface tension between the states

→ Vanishing mass gap (exponential decrease of correlations)

– $\beta < \beta_c$

→ no spontaneous magnetization

→ Unique phase: disordered state

→ Positive mass gap (finite correlation length)

– $\beta = \beta_c$

→ discontinuity of mean energy and magnetization

→ $q+1$ phases

→ Positive surface tension between the phases

→ Positive mass gap

Kotecky Shlosman (1982),...

- $h > 0$ small q large : First order transition

at some $\beta_c(h)$

– $\beta > \beta_c(h)$

→ 1 ordered translation invariant Gibbs state

– $\beta < \beta_c(h)$

→ Unique phase: disordered state

– $\beta = \beta_c(h)$

→ discontinuity of mean energy

→ 2 phases

→ Positive surface tension between the phases

Bakchich, Benyoussef, Laanait (1989),...

Various representations

Sarting from $\omega_{\mathrm{ES}}(\boldsymbol{\sigma}, \boldsymbol{\eta})$

1. Fortuin-Kasteleyn in external field

$$
\begin{aligned}
e^{h\delta_{\sigma_i,1}} &= 1 + (e^h - 1)\delta_{\sigma_i,1} \\
& \sum_{\theta_i=0,1} \delta_{\theta_i,0} + (e^h - 1)\delta_{\sigma_i,1}\delta_{\theta_i,1}
\end{aligned}
$$

   Sum over spin variables ω_{ES}

$$
\begin{aligned}
\omega_{\mathrm{FK}}(\boldsymbol{\eta}, \boldsymbol{\theta}) &= \prod_{\langle i,j\rangle} e^{-\beta\delta_{\eta_{ij},0}}(1 - e^{-\beta})^{\delta_{\eta_{ij},1}} \\
&\times \prod_i e^{-h\delta_{\theta_i,0}}(1 - e^{-h})^{\delta_{\theta_i,1}}\, q^{C(\boldsymbol{\eta}|\boldsymbol{\theta})}
\end{aligned}
$$

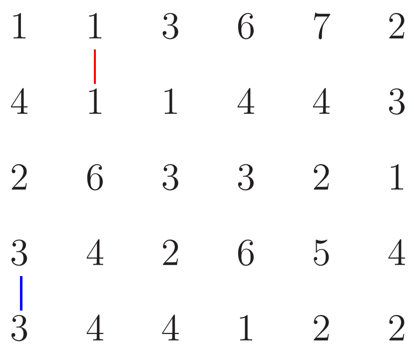
2. Sum directly over spins ω_{ES}

$$
\begin{aligned}
\omega(\boldsymbol{\eta}) &= \prod_{\langle i,j\rangle} e^{-\beta\delta_{\eta_{ij},0}}(1 - e^{-\beta})^{\delta_{\eta_{ij},1}} \\
&\times \prod_i^{C(\boldsymbol{\eta})} (1 + (q-1)e^{-h})^{S_i(\boldsymbol{\eta})}
\end{aligned}
$$

Colored–Edwards–Sokal representation

Write $\delta_{\sigma_i,\sigma_j} = \chi(\sigma_i = \sigma_j = 1) + \chi(\sigma_i = \sigma_j \neq 1)$
Replace edge variables $\eta_{ij} \in \{0, 1\}$ by $n_{ij} \in \{0, 1, 2\}$ (white red blue).
Then

$$\begin{aligned} \omega_{\mathrm{CES}}(\boldsymbol{\sigma}, \boldsymbol{n}) \; = \; & \prod_{\langle i,j \rangle} \Big[e^{-\beta} \delta_{n_{ij},0} \\ & + (1 - e^{-\beta}) \delta_{n_{ij},1} \, \chi(\sigma_i = \sigma_j = 1) \\ & + (1 - e^{-\beta}) \delta_{n_{ij},2} \, \chi(\sigma_i = \sigma_j \neq 1) \Big] \\ & \times \prod_i e^{h \delta_{\sigma_i,1}} \end{aligned}$$



Tri–Clor–Edge representation

Sum over spin variables

$$\begin{aligned}\omega_{\mathrm{TER}}(\boldsymbol{n}) &= \prod_{\langle i,j\rangle} e^{-\beta\delta_{n_{ij},0}}(1-e^{-\beta})^{(\delta_{n_{ij},1}+\delta_{n_{ij},2})} \\ &\quad \times e^{hS_1(\boldsymbol{n})}(q-1)^{C_2(\boldsymbol{n})} \\ &\quad \times (q-1+e^h)^{|\Lambda|-S_1(\boldsymbol{n})-S_2(\boldsymbol{n})}\end{aligned}$$

- $S_1(\boldsymbol{n})$ (resp. $S_2(\boldsymbol{n})$) denotes the number of sites that belong to edges of color 1 (resp. of color 2)
- $C_2(\boldsymbol{n})$ denotes the number of connected components of the set of edges of color 2
- $|\Lambda|$ is the number of sites of the box under consideration

Geometric order parameter

Let $p_\Lambda(i \leftrightarrow j)$ be the probability that the site i is connected to j by a path of edges of color 1.

mass–gap (inverse correlation length)

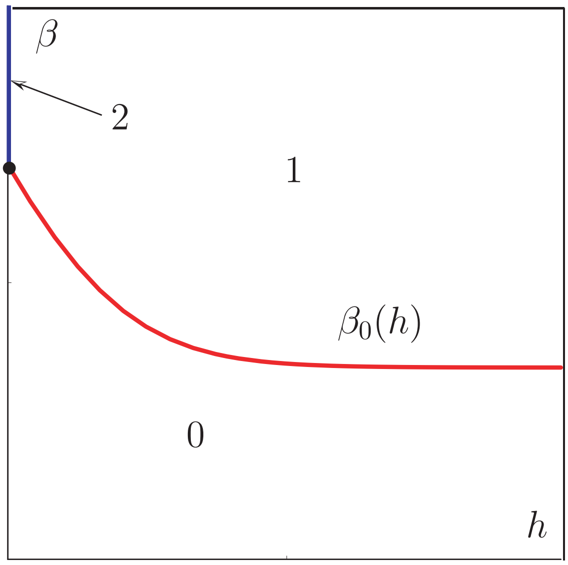
$$m(\beta,h) = -\lim_{|i-j|\to\infty} \frac{1}{|i-j|} \ln \lim_{\Lambda\uparrow\mathbb{Z}^d} p_\Lambda(i\leftrightarrow j)$$

- i and j belong to some line parallel to an axis of the lattice.

Diagram of ground states

- b_a be the value of the Boltzmann weight (in the TER representation) of the ground state configuration of color $a = 0, 1, 2$ per unit site

$$
\begin{aligned}
b_0 &= e^{-\beta d}(q - 1 + e^h) \\
b_1 &= (1 - e^{-\beta})^d e^h \\
b_2 &= (1 - e^{-\beta})^d
\end{aligned}
$$



$$\beta_0(h) = \ln\left[1 + (1 + (q-1)e^{-h})^{1/d}\right]$$

All the ground states coexist at $(0, \beta_0(0))$.
Below $\beta_0(h)$ only the 0–state dominates.
Above $\beta_0(h)$ the 1–state dominates:
it coexists with the 0–state on $\beta_0(h)$ and with the 2–state on the line $h = 0$, $\beta \geq \beta_0(0)$

Analytic results

q is large enough and h not too large

Using Pirogov–Sinaï theory

- the phase diagram of the TER model mimics the diagram of ground state configurations
- the model undergoes a thermodynamic first order phase transition in the sense that the derivative of its free energy with respect to β (or h) is discontinuous at some $\beta_c(h) \sim \beta_0(h)$
- the model exhibits a geometric (first order) transition, in the sense that, on the critical line, the mass gap is discontinuous.

Theorem 1. *Assume $d \geq 2$, q and h such that*

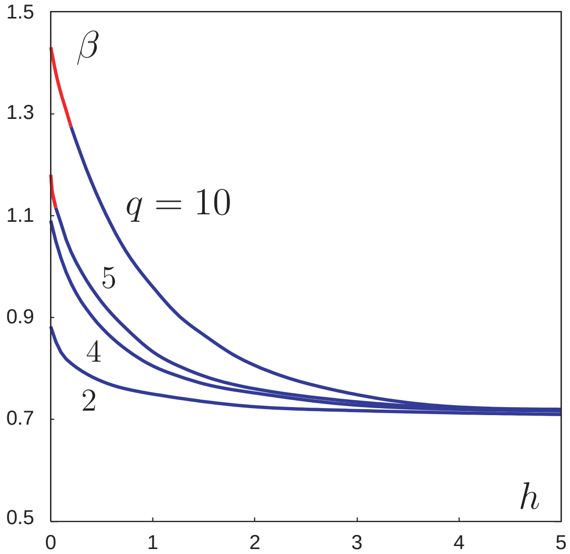
$$c_d(1 + (q-1)e^{-h})^{-1/2d} < 1$$

holds, where c_d is a given number (depending only on the dimension), then there exists a unique $\beta_c(h) = \beta_0(h) + O(1 + (q-1)e^{-h})^{-1/2d})$ such that $m(\beta, h) > 0$ for $\beta \leq \beta_c(h)$ and $m(\beta, h) = 0$ for $\beta > \beta_c(h)$.

Since the free energies of Potts model and of the TER model are the same, the critical lines coincides with that of Potts model.

Numerical simulations

Generalization of the Swendsen–Wang algorithm inherited from colored Edwards–Sokal model.



$d = 2$.

1. For $q \leq 4$:

   – a whole geometric transition line for which

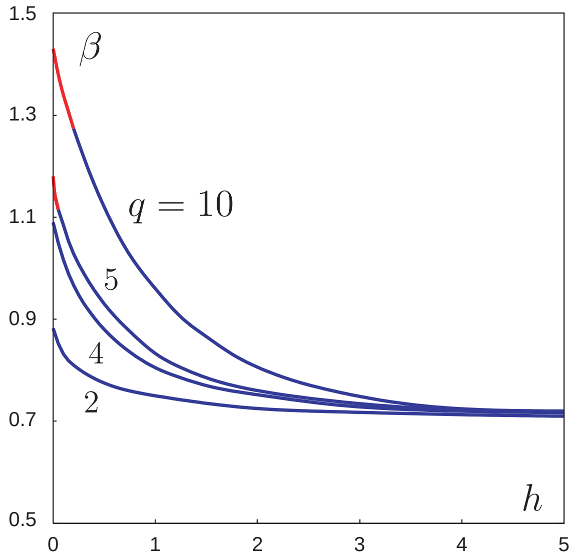
$$m(\beta, h) > 0 \qquad \text{when } \beta \leqslant \beta_c(h)$$
$$m(\beta, h) = 0 \qquad \text{when } \beta > \beta_c(h)$$

   – The mass gap is continuous at $\beta_c(h)$

   – For $\beta \leqslant \beta_c(h)$ the mean cluster sizes remain finite

   – For $\beta > \beta_c(h)$ the size of 1–edge clusters diverges.

   – The mean energy as well as the magnetisation do not show any singular behavior



2. For $q \geq 5$: some critical h_c appears

   – the transition becomes first order when $h < h_c$ in accordance with the previous analytic results

   – both the mass gap and the mean energy exhibit discontinuities at $\beta_c(h > h_c)$.

   – However when $h \geq h_c$, the scenario is the same as for $q \leq 4$.

Numerics are in accordance with theory for vanishing and infinite fields: $\beta_c(0) = \ln(1 + \sqrt{q})$ and $\beta_c(\infty) = \ln 2$.

Description of the algorithm

inherited from colored Edwards–Sokal model.

$$\begin{aligned}\omega_{\mathrm{CES}}(\boldsymbol{\sigma},\boldsymbol{n}) \;=\; & \prod_{\langle i,j\rangle}\Big[e^{-\beta}\delta_{n_{ij},0} \\ & + (1-e^{-\beta})\delta_{n_{ij},1}\,\chi(\sigma_i=\sigma_j=1) \\ & + (1-e^{-\beta})\delta_{n_{ij},2}\,\chi(\sigma_i=\sigma_j\neq 1)\Big] \\ & \times \prod_i e^{h\delta_{\sigma_i,1}}\end{aligned}$$

1. Given a spin configuration:
   put between any two neighbouring spins of the same color:
   - an edge colored 0 with probability $e^{-\beta}$
   - w.p. $1 - e^{-\beta}$, an edge colored 1 if these spins are of colour 1, and coloured 2 otherwise.

   When two neighbouring spins disagree, the corresponding edge is colored 0.

2. Starting from an edge configuration, a spin configuration is constructed as follows.
   - Isolated sites (endpoints of 0–bonds only) are coloured 1 w.p. $e^h/(q-1+e^h)$ and coloured $c \in \{2, ..., q\}$ w.p. $1/(q-1+e^h)$.
   - Non–isolated sites are colored 1 (w.p. 1) if they are endpoints of 1–bonds and colored $c \in \{2, ..., q\}$ w.p. $1/(q-1)$.

Scheme of proof

Dilute partition function

partition function

$$Z_a = \sum_{\boldsymbol{n}} \omega_{\mathrm{TER}}(\boldsymbol{n})\chi^a(\boldsymbol{n})$$

up to a boundary term

$$Z_a(\Lambda) = \sum_{\boldsymbol{n}} \prod_{i\in\Lambda} \omega_i(\boldsymbol{n}) q^{C_2(\boldsymbol{n})-\delta_{a,2}} \prod_{i\in\partial\Lambda} \prod_{j\sim i} \delta_{n_{ij},a}$$

sum is over all configurations $\boldsymbol{n} = \{n_{ij}\}_{ij\cap\neq\emptyset}$,
$\partial\Lambda$ is the boundary of Λ
$i\sim j$ means that i and j are n.n.

$$\begin{aligned} \omega_i(\boldsymbol{n}) &= (1-e^{-\beta})^{(\delta_{n_{ij},1}+\delta_{n_{ij},2})/2} e^{-\beta\delta_{n_{ij},0}/2} e^{h\chi(i\in``1'')} \\ &\quad \times (q-1+e^h)^{\prod_{j\sim i}\delta_{n_{ij},0}} \end{aligned}$$

$\chi(i\in``1'')$ means that the site i belongs to some edge of color 1.

Contours (signed)

- $\boldsymbol{n}$ a configuration on : $E(\Lambda) = \{\langle i,j\rangle \cap \Lambda \neq \emptyset\}$
- $i\in\Lambda$ is called correct if for all $j\sim i$, n_{ij} take the same value
- $i\in\Lambda$ is called incorrect if for all $j\sim i$, n_{ij} take the same value otherwise
- $\Gamma = \{\mathrm{Supp}\Gamma, \boldsymbol{n}(\Gamma)\}$ is called contour if :
  SuppΓ) is a maximal connected subset of incorrect sites
  $\boldsymbol{n}(\Gamma)$ the restriction of $\boldsymbol{n}$ to E(Supp Γ)

Contour expansions

over external contours

$$Z_a(\Lambda) = \sum_{\theta=\{\Gamma_1,\dots,\Gamma_n\}_{\text{ext}}} b_a^{|\text{Ext}_\Lambda \theta|} \times \prod_{k=1}^{n} \rho(\Gamma_k) \prod_{m=0,1,2} Z_m(\text{Int}_m \Gamma_k)$$

where

$$\rho(\Gamma) = \prod_{i \in \text{supp}\Gamma} \omega_i(\boldsymbol{n}_\Gamma) q^{C(\boldsymbol{n}_\Gamma) - \delta_{a,2}}$$

over compatible contours

$$Z_a(\Lambda) = b_a^{|\Lambda|} \sum_{\{\Gamma_1,\dots,\Gamma_n\}_{\text{comp}}} \prod_{k=1}^{n} z_a(\Gamma_k)$$

with activities

$$z_a(\Gamma) = \rho(\Gamma) b_a^{-|\text{supp}\Gamma|} \prod_{m \neq a} \frac{Z_m(\text{Int}_m \Gamma)}{Z_a(\text{Int}_m \Gamma)}$$

Peierls' estimate

$$\rho(\Gamma)(\max_{a=0,1,2} b_a)^{-|\mathrm{Supp}\Gamma|} \leq e^{-\tau|\mathrm{Supp}\Gamma|} \tag{1}$$

where

$$e^{-\tau} = (1 + (q-1)e^{-h})^{-1/2d}$$

First notice that an incorrect site i is either of color 1 or of color 2. In the first case one has $\sum_{j\sim i}(\delta_{n_{ij},0} + \delta_{n_{ij},1}) = 2d$, so that $\omega_i(\boldsymbol{n}_\Gamma)/b_1 = (e^\beta - 1)^{-(\sum_{j\sim i}\delta_{n_{ij},0})/2}$ implying

$$\omega_i(\boldsymbol{n}_\Gamma)/\max_{a=0,1,2} b_a \leq (1 + (q-1)e^{-h})^{-(\sum_{j\sim i}\delta_{n_{ij},0})/2d}$$

Since $1 \leq \sum_{j\sim i}\delta_{n_{ij},0} \leq 2d-1$, each incorrect site of color 1 gives at most a contribution $e^{-\tau}$ to the L.H.S. of (1). In the second case, one has $\sum_{j\sim i}(\delta_{n_{ij},0} + \delta_{n_{ij},2}) = 2d$, so that $w_i(\boldsymbol{n}_\Gamma)/b_2 = (e^\beta - 1)^{-(\sum_{j\sim i}\delta_{n_{ij},0})/2}$ implying

$$\omega_i(\boldsymbol{n}_\Gamma)/\max_{a=0,1,2} b_a \leq (q - 1 + e^h)^{-(\sum_{j\sim i}\delta_{n_{ij},0})/2d}$$

We then use again that $1 \leq \sum_{j\sim i}\delta_{n_{ij},0} \leq 2d-1$ and that $C_2(\boldsymbol{n}_\Gamma) \leq \sum_{i\in\mathrm{Supp}\Gamma}\chi(1 \leq \delta_{n_{ij},2})/2^{\sum_{j\sim i}\delta_{n_{ij},2}}$, to obtain that each incorrect site of color 2 gives at most a contribution $(e^h + q - 1)^{-1/2+1/2d} \leq e^{-\tau}$ to the L.H.S. of (1).

Going on with PS (with Zahradnick formulation)

Peierls' estimate with small $e^{-\tau}$ (assumption of Thm gives by PS theory good control of the system

Truncated activity

$$z_a'(\Gamma)=\begin{cases} z_a(\Gamma) & \text{if } z_a(\Gamma)\le e^{-(\tau-\tau_0)|\mathrm{Supp}\,\Gamma|} \\ e^{-(\tau-\tau_0)|\mathrm{Supp}\,\Gamma|} & \text{otherwise}\end{cases}$$

τ_0 some numerical constant

Stable contours: call a contour stable if $z_a(\Gamma)=z_a'(\Gamma)$

Truncated partiton functions: leave out unstable contours i.e. take the activities $z_a'(\Gamma)$ in the expansion:

$$Z_a(\Lambda)=b_a^{|\Lambda|}\sum_{\{\Gamma_1,\ldots,\Gamma_n\}_{\mathrm{comp}}}\prod_{k=1}^{n} z_a'(\Gamma_k)$$

metastable free energies:

$$f_a^{\mathrm{met}}(\beta,h)=-\lim_{\Lambda\uparrow\mathbb{Z}^d}(1/|\Lambda|)\ln Z_a'(\Lambda)$$

Observe

$$f_a^{\mathrm{met}}(\beta,h)=-\ln b_a+\text{Correction}$$

Correction $=O(e^{-\tau})$, because they are free energies of contour models which can be controlled by convergent cluster expansions:

Stable boundary conditon : bc is called stable if

$$f_a^{\mathrm{met}}(\beta,h)=\min_{m=0,1,2} f_a^{\mathrm{met}}(\beta,h)$$

If the bc a is stable then all a–contours are stable

Thus: as a standard result of Pirogov-Sinai theory, one gets that the phase diagram of the system is a small perturbation of the diagram of ground state configurations.

- unique point $\beta_c(0)$ given by the solution of

  $$f_0^{\mathrm{met}}(\beta,h)=f_1^{\mathrm{met}}(\beta,h)=f_2^{\mathrm{met}}(\beta,h)$$

  for which all contours are stable and such that $Z_a(\Lambda)=Z'_a(\Lambda)$ for $a=0,1,2$

- line $\beta_c(h)$ given by the solution of

  $$f_0^{\mathrm{met}}(\beta,h)=f_1^{\mathrm{met}}(\beta,h)$$

  when $h>0$ and such that, $Z_a(\Lambda)=Z'_a(\Lambda)$ for $a=0,1$

- $\beta<\beta_c(h)$ one has $Z_0(\Lambda)=Z'_0(\Lambda)$

- $\beta>\beta_c(h)$ one has $Z_1(\Lambda)=Z'_1(\Lambda)$

- For $h=0$ and $\beta\geq\beta_c(0)$, one has in addition $Z_2(\Lambda)=Z'_2(\Lambda)$

End of proof: dicontinuity of mass–gap

1. If one impose that the site i is connected to j by a path made up of edges of color 1, then under the boundary condition 0, there exists necessarily an external contour that encloses both the sites i and j.
   Probability of external contours Γ decays like $(c_0 e)^{-\tau|\mathrm{Supp}\Gamma|}$ when the 0–contours are stable, i.e. when $Z_0(\Lambda) = Z_0'(\Lambda)$

   – $p_\Lambda(i \leftrightarrow j) \leq (\mathrm{Ctee}^{-\tau})^{|i-j|}$ when $\beta \leq \beta_c(h)$

2. Under the bc. 1, by Peierls type arguments, the probability that the site i is not connected to j can be bounded from above by a small number $O(e^{-\tau})$ when $Z_1(\Lambda) = Z_1'(\Lambda)$
   $\Rightarrow$

   $$p_\Lambda(i \leftrightarrow j) \geqslant 1 - 0(e^{-\tau}) \text{ for } \beta \geq \beta_c(h)$$

   $\Rightarrow$ Under the bc. 0

   $$p_\Lambda(i \leftrightarrow j) \geqslant 1 - 0(e^{-\tau}) \text{ for } \beta > \beta_c(h)$$

“Kertèsz line in Curie–Weiss Potts model (Potts model on the complete graph)

Erdös–Renyi Random Graph:

$G_{n,p}$

- Consider n vertices
- Put a bond between any two vertices with probability p (and no bond w.p. $(1-p)$

Take $p = \frac{c}{n}$

Phase transition:

→ if $c < 1$: with high probability only components of size $O(\ln n)$ (or less)

→ If $c \geqslant 1$: A component of size $O(n)$ appears (Giant component) and the other ones have size $O(\ln n)$ (w.h.p.)

Only transition in the topology: nothing happens for the mean occupation number of bonds

Curie–Weiss Potts model

n vertices

$\sigma_i \in \{1, \ldots, q\}$ $\qquad\qquad i = 1, \ldots, n$

Boltzmann weight

$$\omega_{\text{CWPotts}}(\boldsymbol{\sigma}) \;=\; \prod_{i<j} e^{\frac{\beta}{n}(\delta_{\sigma_i,\sigma_j}-1)}$$

- For $q = 2$: Second order phase transition at $\beta_c = 2$
- For $q \geqslant 3$: First order transition at $\beta_c = 2\frac{q-1}{q-2}\ln(q-1)$

The transition is characterized by the order parameter s solution of the mean–field equation

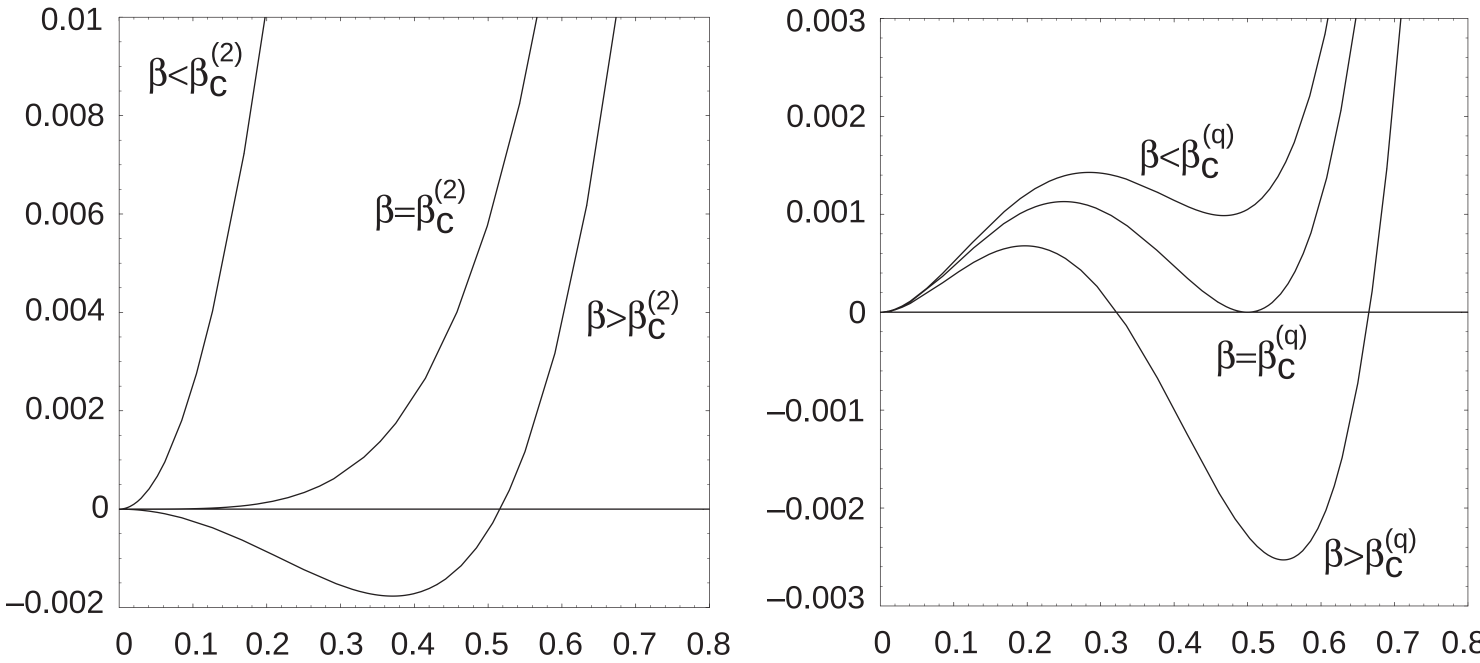
$$s = \frac{e^{\beta s} - 1}{e^{\beta s} + q - 1}$$

- For $q = 2$
  - $\beta \leqslant \beta_c$ $\qquad s = 0$
  - $\beta > \beta_c$ $\qquad s > 0$
- For $q \geqslant 3$
  - $\beta < \beta_c$ $\qquad s = 0$
  - $\beta = \beta_c$ $\qquad s_c = \frac{q-2}{q-1}$
  - $\beta > \beta_c$ $\qquad s > s_c$

Wu (1982),...

$f_{\text{can}}^{\text{CWPotts}}$: canonical free energy of the model with fixed density of colors ρ_1, ..., ρ_q (can be explicitely computed

Minima of $f_{\text{can}}^{\text{CWPotts}}$ parametrized $s \in [0,1]$, and components of the vector $(\rho_1, ..., \rho_q)$ permutations of the q values: $\frac{1}{q}(1+(q-1)s), \frac{1}{q}(1-s), ..., \frac{1}{q}(1-s)$



- $q=2$
  - $\rightarrow$ $\beta \leqslant \beta_c$ unique minimizer $(s=0)$
  - $\rightarrow$ $\beta > 2$ two minima $\{\frac{1}{2}+s_0, \frac{1}{2}-s_0\}$ and $\{\frac{1}{2}-s_0, \frac{1}{2}+s_0\}$ with s_0 sol. of MFE
- $q \geqslant 3$
  - $\rightarrow$ $\beta < \beta_c$ unique minimizer $(s=0)$
  - $\rightarrow$ $\beta = \beta_c$ $q+1$ minimizers: $(s=0)$ and the permutations of the above vector, with $s=s_c$
  - $\rightarrow$ $\beta > \beta_c$ q minimizers: permutations of the above vector with $s > s_c$

Edwads-Sokal and Fortuin–Kasteleyn representation of CW Potts model

$$\omega_{\mathrm{ES}}^{\mathrm{CW}}(\boldsymbol{\sigma},\boldsymbol{\eta}) = \prod_{i<j} e^{-\beta/n}\,\delta_{\eta_{ij},0} + (1-e^{-\beta/n})\delta_{\eta_{ij},1}\delta_{\sigma_i,\sigma_j}$$

$$\omega_{\mathrm{FK}}^{\mathrm{CW}}(\boldsymbol{\eta}) = \prod_{i<j} [e^{-(\beta/n)\,\delta_{\eta_{ij},0}}(1-e^{-\beta/n})^{\delta_{\eta_{ij},1}}]\,q^{C(\boldsymbol{\eta})}$$

For $q=1$, Erdös–Renyi random graph $G_{n,p}$ with

$$p = 1 - e^{-\beta/n} \simeq \frac{\beta}{n}$$

Results for FK representation (Bollobas, Grimmet Janson (1996)

- $q = 1$ classical ER transition from No Giant to Giant component at $\beta_c = 1$
- $q = 2$
  - → Transition from No Giant to Giant component at $\beta_c = 2$
  - → Second order transition at $\beta_c = 2$ in the mean occupation number of bonds (corresponding to the second order transition of CWPotts)
- $q = 3$
  - → Transition from No Giant to Giant component at $\beta_c = 2$
  - → First order transition at $\beta_c = 2\frac{q-1}{q-2}\ln(q-1)$ in the mean occupation number of bonds (corresponding to the first order transition of CWPotts)

Able to study all real (positive) values of q

Add a magnetic field

$$\omega_{\mathrm{CWES}}(\boldsymbol{\sigma},\boldsymbol{\eta}) = \prod_{i<j} [e^{-\beta/n}\delta_{\eta_{ij},0} + (1-e^{-\beta/n})\delta_{\eta_{ij},1}\delta_{\sigma_i,\sigma_j}] \times \prod_i e^{h\delta_{\sigma_i,1}}$$

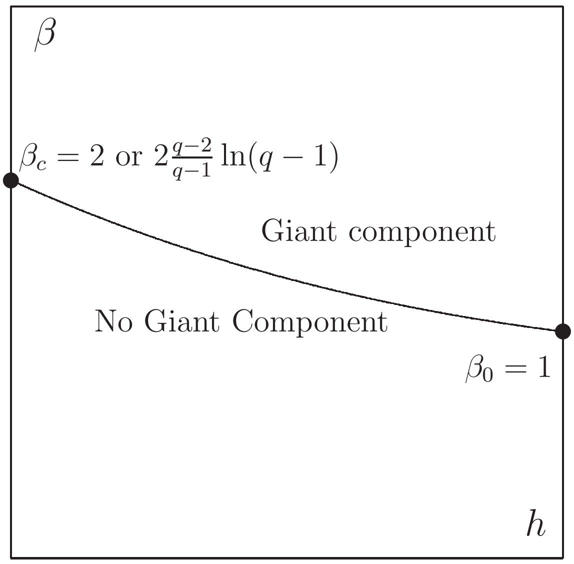
- $h=\infty$ all $\sigma_i = +1$

$$\omega_{\mathrm{CWES}}(\boldsymbol{\eta}) = \prod_{\langle i,j\rangle} [e^{-\beta/n}\delta_{\eta_{ij},0} + (1-e^{-\beta/n})\delta_{\eta_{ij},1}]$$

$\rightarrow$ ER random graph $G_{n,p}$ with $p = 1 - e^{-\beta/n}$

$\rightarrow$ ER transition at $\beta_c = \ln 1$

Consider CES representation and TER representation on the complete graph



What we expect (for $h > 0$)

- $q = 2$ no transition in mean occupation number of 1–bonds
- $q \geqslant 3$
  - appearance of a critical h_c
    - $\rightarrow$ $h < h_c$ First order transition in the mean occupation number of 1–bonds
    - $\rightarrow$ $h \geqslant h_c$ No transition in MON

Why: because with $h > 0$ the Mean field equation reads (Biskup Chayes Crawford 2006)

$$s = \frac{e^{\beta s + h} - 1}{e^{\beta s + h} + q - 1}$$

- $q = 2$ only one solution
- $q \geqslant 3$
  - $\rightarrow$ for $h < h_c = \ln q - \frac{2(q-2)}{q}$ two solutions at some $\beta_c(h)$
  - $\rightarrow$ for $h \geqslant h_c$ one solution

$\beta_c(h_c) = 4\frac{q-1}{q}$