$\mathcal{LU}$-factorization and probabilities

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Introduction

- $\mathcal{LU}$-factorization of $\mathcal{A}$:

$$\mathcal{A} = \mathcal{LU}$$

  where
  $\mathcal{L}$ is Lower triangular
  $\mathcal{U}$ is Upper triangular.
  For unicity we need to precise that $\mathcal{L}$ have diagonal entries equal to 1.
- Our subject : $\mathcal{A} = I - P$.

Questions

- The "developped" $\mathcal{LU}$-factorization allways exist.
- When the "true" $\mathcal{LU}$-factorization exist ?
- When the $\mathcal{LU}$-factorization is unic ?
- When the $\mathcal{LU}$-facrorization is associative : $(\mathcal{LU})f = \mathcal{L}(\mathcal{U}f)$?
- When we $\mathcal{LU}$-facrorization is commutative : $\mathcal{LU} = \mathcal{UL}$?
- Probabilistic interpretation of all these...

Markov staff

- P sub-markovian ($P1 \leq 1$) on E denumerable.
- P considerated as a transition kernel :

$$\mathbf{E}_x[1_{\{X_0=x_o\,,\,X_1=x_1\,,\,\dots\,,\,X_t=x_t\}}] = I(x,x_o)P(x_o,x_1)...P(x_{t-1},x_t)$$

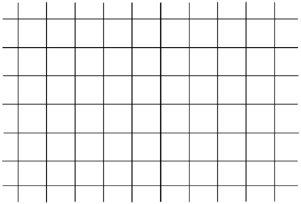
- When $P1 \neq 1$, the markov process can die.
- the potential kernel relative to P by :

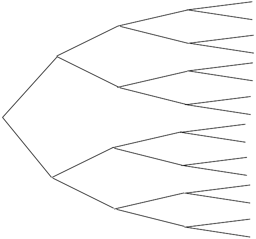
$$U(x,y) = \sum_{t=0}^{\infty} P^t(x,y) = \mathbf{E}_x \sum_t 1_{\{X_t=y\}}$$

(no links with $\mathcal{U}$).

An altitude

- $\mathfrak{a} : E \mapsto \mathbb{R}$, $x \preceq y \;\Leftrightarrow\; \mathfrak{a}(x) \leq \mathfrak{a}(y)$
- Complicate examples :

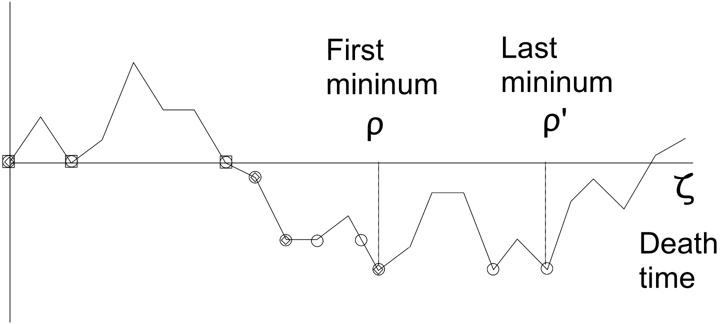




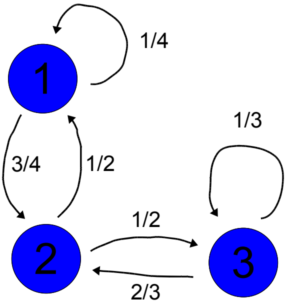
- Simple examples : $E \subset \mathbb{Z}$ and " $\preceq$ " $=\leq$.

Descending processes

- $\gimel^o$ goes from a state to the following state at the same altitude until X cross under X_0.
- $\gimel'$ goes from a state to the following state at an inferior altitude.
- $\gimel$ goes from a state to the following state at a strictly inferior altitude.



- K^o, K', K their transition kernel.
  V^o, V', V their potential kernel.
- Example



$$K^o = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

$$K' = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 3/4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

K as series

- Let $A \subset E$. We denote by

$$P_A(x,y) = P(x,y)1_{\{x\in A\}}1_{\{y\in A\}} \qquad \text{and} \qquad U_A = \sum_t P_A^t$$

with $A = \{\succeq x\} = \{y : y \succeq x\}$:

$$P_{\succeq x}(x,y) = P(x,y)1_{\{y\succeq x\}} \qquad \text{and} \qquad U_{\succeq x} = \sum_t P_{\succeq x}^t$$

- We have

$$\begin{aligned}
K(x,y) &= U_{\succeq x}P(x,y)1_{\{x\succ y\}} \\
K'(x,y) &= PU_{\succ x}P(x,y)1_{\{x\succeq y\}} + P(x,y)1_{\{x\succeq y\}} \\
K^o(x,y) &= PU_{\succ x}P(x,y)1_{\{x\sim y\}} + P(x,y)1_{\{x\sim y\}} \\
V(x,y) &= U_{\succ y}P(x,y) + 1_{\{x=y\}} \\
V'(x,y) &= U_{\succeq y}(x,y) \\
V^o(x,y) &= U_{\succeq y}(x,y)1_{\{x\sim y\}}
\end{aligned}$$

Sweveling

- Kernels K, V... are functions of P

$$K_{[P]}(x,y) = U_{\succeq x}P(x,y)1_{\{x \succ y\}} = \sum_a \sum_n P^n_{\succeq x}(x,a)P(a,y)1_{\{x \succ y\}}$$
$$V_{[P]}(x,y) = U_{\succ y}P(x,y) + 1_{\{x=y\}}$$

- We define

$$K^{\curlyvee}(x,y) = K_{[P^T]}(y,x)$$
$$V^{\curlyvee}(x,y) = V_{[P^T]}(y,x)$$

Those double transpositions give :

$$K^{\curlyvee}(x,y) = PU_{\succeq y}(x,y)1_{\{x \prec y\}}$$
$$V^{\curlyvee}(x,y) = PU_{\succ x}(x,y) + 1_{\{x=y\}}$$

$\mathcal{LU}$-factorizations

- The developped one :

$$K' + K^{\curlyvee} + P = K^{\curlyvee} K' \qquad \text{always true}$$

- The true one :

$$(I - P) = (I - K^{\curlyvee})(I - K') \qquad \text{when } K^{\curlyvee} < \infty$$

- The three factors one :

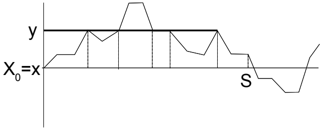
$$(I - P) = (I - K^{\curlyvee})(I - K^{o})(I - K) \qquad \text{when } K^{\curlyvee} < \infty$$

- The inverse one :

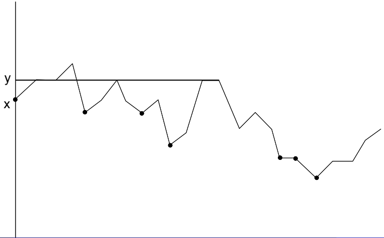
$$U = V' V^{\curlyvee} = V V^{o} V^{\curlyvee} \qquad \text{always true}$$

Proof of the "inverse" factorization

- $$V^{\Upsilon}(x,y) = \mathbf{E}_x \sum_{t=0}^{S} 1_{\{X_t = y\}}$$

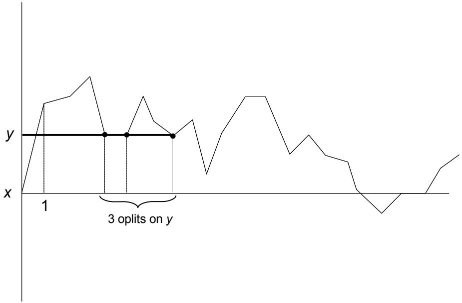


- $$V' V^{\Upsilon}(x,y) = \mathbf{E}_x \sum_{t \in \bullet} \left(\sum_{t=0}^{S} 1_{\{X_t = y\}} \right) \circ \theta_t$$



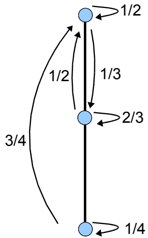
Interpretation of $K^\curlyvee$

- $t \in \{\text{oplit on } y\} \;\Leftrightarrow\; X_t = y \succ X_0$ and $X \succeq y$ on $[1, t]$.



- $$K(x, y) = PU_{\succeq y}(x, y) 1_{\{x \prec y\}} = \mathbf{E}_x[\sharp\{\text{oplit on } y\}]$$

Existence

- **Theorem :** $\mathcal{LU}$-factorization is possible if (and only if) there is no state which is :
  1/ recurrent 2/ undescendable 3/ Reacheable from below.
- E finite : these conditions just depend on the graph of P.
- 

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \\ 3/4 & 0 & 1/4 \end{pmatrix}$$

$$K' = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \quad K^{\curlyvee} = \begin{pmatrix} 0 & 0 & 0 \\ 2/3 & 0 & 0 \\ 3/2 & \infty & 0 \end{pmatrix}$$

$K^{\curlyvee} K' + P = K' + K^{\curlyvee}$ Yes.

$I - P = (I - R^{\curlyvee})(I - R')$ impossible....

Known theorem on M-matrices

- M-matrix is

$$A = \lambda I - Q \qquad \text{with } \rho(Q) \leq \lambda$$

- If h is an egein vector associate to $\rho(Q)$, then

$$\frac{1}{\lambda}\frac{h(y)}{h(x)}Q(x,y) \text{ is sub-Markovian and so } \frac{1}{\lambda}\frac{h(y)}{h(x)}A(x,y) \text{ is a generator}$$

- **Theorems :** For a M-matrix, $\mathcal{LU}$-factorization is possible :
  If it is inversible [Fiedler and Ptak, 1962]
  If it is irreductible [Kuo, 1977]
  Iff no state is : recurrent and undescendable and reacheable from below. [Varga, Cai, 1981]

Known theorem on stochastics matrices

- **Theorems :** For generator $I - P$, $\mathcal{LU}$-factorization is possible :
  if P is irreductible, recurrent, on finite E [Grassman 1987]
  if P is irreductible, recurrent, on $\mathbb{N}$ [Heyman 1995]

Commutativity

- Let E be a semigroup, P and $\preceq$ be invariant by translation :

$$P(x,y) = P(x+z, y+z) \qquad\qquad x \preceq y \;\Leftrightarrow\; x+z \preceq y+z$$

- K' and $K^{\curlyvee}$ are also invariant by translation and

$$I - P = (I - K^{\curlyvee})(I - K') = (I - K')(I - K^{\curlyvee})$$

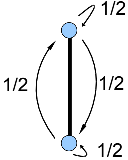
Case $E = \mathbb{Z}$ is better known under the name of "Wiener-Hopf factorization"

Unicity

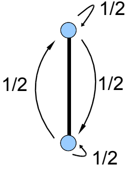
- **Theorem** $E = \mathbb{N}$. Fix P. Suppose

$$R' + R^{\curlyvee} = P + R^{\curlyvee} R' \qquad \text{with } ....$$

Then $R' = K'$ and :
If j is transient, descendable Then $R^{\curlyvee}_{ij} = K^{\curlyvee}_{ij} \in \mathbb{R}_+$
If j is rec., undescendable, reachable from $i < j$ Then $R^{\curlyvee}_{ij} = K^{\curlyvee}_{ij} = \infty$
If j is rec., undescendable, not reach. from $i < j$ Then $R^{\curlyvee}_{ij}$ can be anythink



$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{pmatrix} =$$



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & b & -1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Associativity

- Suppose $K^{\curlyvee} < \infty$.

$$[(I - K^{\curlyvee})(I - K')]1(x) \leq (I - K^{\curlyvee})[(I - K')1](x)$$

The difference between them is

$$\mathbf{P}_x[X \succ x \text{ on } [1, \infty[\;\; , \;\; X_1 \neq \dagger \;\; , \;\; \rho' \circ \theta_1 = \infty] = 0$$

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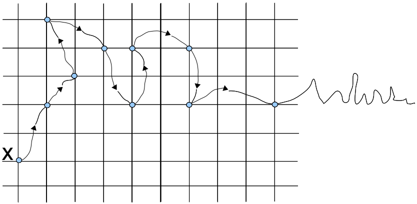
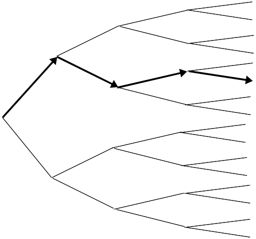
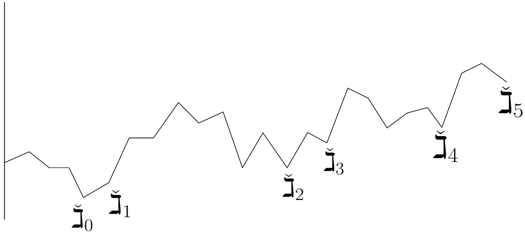




h-transform of K^{Υ}

- $(I-P)h \geq 0 \;\Rightarrow\; (I-K^{\Upsilon})\big[(I-K')h\big] \geq 0$
- $k' := (I-K')1 = \mathbf{E}_x\big(X \succ x \text{ on } [1,\infty[\big)$.

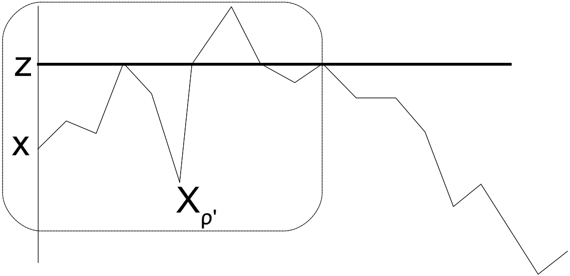
$$\check{K}(x,y) := \frac{k'(y)}{k'(x)} K^{\Upsilon}(x,y) \text{ is sub-Markovian}$$



Recall that : $K^{\Upsilon} = K^{T}_{[P^T]} \cdots$

Minima

- Suppose $U(x,z) \in]0,\infty[$.
  Let $\mathbf{E}_{x \triangleright z}$ be the law of X started at x, killed the last time it goes in z.



- We have

$$\mathbf{E}_{x \triangleright z}[X_{\rho'} = a] U(x,z) = V'(x,a) V^{\curlyvee}(a,z)$$

- Suppose X dies :

$$\mathbf{E}_x[X_{\rho'} = a] = V'(x,a) \sum_z V^{\curlyvee}(a,z) P(z,\dagger)$$

Algorithm

```
P=[0.2 0.3 0.1 0.2 ; 0.4 0.1 0.2 0.1 ; 0.1 0 0.2 0.4 ;
0.5 0.1 0.1 0.2]

 P  =
    0.2    0.3    0.1    0.2
    0.4    0.1    0.2    0.1
    0.1    0.     0.2    0.4
    0.5    0.1    0.1    0.2

Un=[1; 1; 1; 1]; I=eye(4,4);
Pmourir=Un-P*Un

 Pmourir  =
    0.2
    0.2
    0.3
    0.1
```

```
[Lower,Upper]= lu(I-P);
Kpivote= I-Lower
Kprime=I-Upper

 Kpivote  =
    0.       0.           0.           0.
    0.5      0.           0.           0.
    0.125    0.05         0.           0.
    0.625    0.3833333    0.3333333    0.
 Kprime  =
    0.2    0.3     0.1      0.2
    0.     0.25    0.25     0.2
    0.     0.      0.225    0.435
    0.     0.      0.       0.5466667


Vpivote=Lower^(-1)
Vprime=Upper^(-1)

 Vpivote  =
    1.           0.      0.           0.
    0.5          1.      0.           0.
    0.15         0.05    1.           0.
    0.8666667    0.4     0.3333333    1.
 Vprime  =
    1.25    0.5          0.3225806    1.0815939
    0.      1.3333333    0.4301075    1.0009488
    0.      0.           1.2903226    1.2381404
    0.      0.           0.           2.2058824
```

```
U=Vprime*Vpivote
verif=(I-P)^(-1)

 U  =
    2.4857685    0.9487666    0.6831120    1.0815939
    1.5986717    1.7552182    0.7637571    1.0009488
    1.2666034    0.5597723    1.7030361    1.2381404
    1.9117647    0.8823529    0.7352941    2.2058824
 verif  =
    2.4857685    0.9487666    0.6831120    1.0815939
    1.5986717    1.7552182    0.7637571    1.0009488
    1.2666034    0.5597723    1.7030361    1.2381404
    1.9117647    0.8823529    0.7352941    2.2058824


Toto=(Vpivote*Pmourir*ones(1,4))';
LoiDuMin=Vprime.*Toto

 LoiDuMin  =
    0.25    0.15    0.1096774    0.4903226
    0.      0.4     0.1462366    0.4537634
    0.      0.      0.4387097    0.5612903
    0.      0.      0.           1.
```