Large Deviations for Statistics of Jacobi Processes

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Sketch of talk

- Jacobi process = Unique strong solution on $[-1,1]$ of SDE

$$\begin{cases} dY_t = \sqrt{1-Y_t^2}dW_t + (bY_t + c)dt \\ Y_0 = y_0 \end{cases}$$

- Aim: derive a LDP for estimate of b in ultraspherical case: $c = 0$
- Question: handable form for the semi-group density p ?

Classification of Bakry & Mazet

- $\mu << \lambda$ measure on I interval
  $\exists\lambda > 0\,; \quad \int_I e^{\lambda|y|}\mu(dy) < \infty \Rightarrow$ orthonormal base $(R_n)_n$ of polynomials in $L^2(I)$

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$$\forall n\,, P_t R_n = e^{-\lambda_n t} R_n$$

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$$\Rightarrow L = (Ax^2 + Bx + C)\frac{d^2}{dx^2} + (ax + b)\frac{d}{dx}$$

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- $$L = (1 - x^2)\frac{d^2}{dx^2} + (\beta - \gamma - (\beta + \gamma + 2)x)\frac{d}{dx}$$
  Jacobi semi-group ; R_n= Jacobi

Caracterisation :

- Infinitesimal generator

$$L = (1-x^2)\frac{\partial^2}{\partial^2 x} + (px+q)\frac{\partial}{\partial x}, \qquad x \in [-1,1]$$

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- $$LP_n^{\alpha,\beta} = -n(n+\alpha+\beta+1)P_n^{\alpha,\beta}$$

  and $p = -(\beta+\alpha+2)$ and $q = \beta-\alpha$ Jacobi polynomial of parameters $\alpha, \beta > -1$

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1, \alpha+1; \frac{1-x}{2}\right)$$

Mehler type formula ?

(Wong 1964)

$$p_t(x,y) = \left(\sum_{n\geq 0} (R_n)^{-1} e^{-\lambda_n t} P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y)\right) W(y), \qquad x, y \in [-1, 1]$$

$$\lambda_n = n(n + \alpha + \beta + 1)$$

B Beta function,

$$W(y) = \frac{(1-y)^\alpha (1+y)^\beta}{2^{\alpha+\beta+1} B(\alpha+1, \beta+1)}, \quad R_n = ||P_n^{\alpha,\beta}||^2_{L^2([-1,1], W(y)dy)}$$

O-U and squared O-U cases: $\lambda_n = n$
Jacobi λ_n quadratic $\longrightarrow$ computation of p_t ?

Subordinated process

- $B_t^{\mu} := B_t + \mu t,\ \mu > 0,$

$$T_t^{\mu,\delta} = \inf\{s > 0; \quad B_s^{\mu} = \delta t\}, \qquad \delta > 0.$$

Subordinated process

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$$T_t^{\mu,\delta} = \inf\{s > 0; \quad B_s^\mu = \delta t\}, \qquad \delta > 0.$$

- Martingale methods, $t > 0,\ u \geq 0,$

$$\mathbb{E}(e^{-uT_t^{\mu,\delta}}) = e^{-t\delta(\sqrt{2u+\mu^2}-\mu)}$$

density of T_t: $t > 0$

$$\nu_t(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t \mu} s^{-3/2} \exp\left(-\frac{1}{2}(\frac{t^2\delta^2}{s} + \mu^2 s)\right) \mathbf{1}_{\{s>0\}}$$

Mehler type formula

Consider subordinated process $Y_{T_t^{\mu,\delta}}$ of semi-group density q_t
Fix δ, μ and write $\lambda_n = (n+\gamma)^2 - \gamma^2$,

$$\begin{aligned} q_t(x,y) &= \int_0^\infty p_s(x,y)\nu_t(s)ds \\ &= W(y)\sum_{n\geq 0}(R_n)^{-1}\mathbb{E}(e^{-\lambda_n T_t^{\mu,\delta}})P_n^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y) \\ &= W(y)\sum_{n\geq 0}(R_n)^{-1}e^{-nt}P_n^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y) \end{aligned}$$

Inverse Laplace

- Besides

$$q_t(x,y) = \frac{t\, e^{\gamma t}}{2\sqrt{\pi}} \int_0^\infty p_s(x,y)\, s^{-3/2} e^{-\gamma^2 s}\, e^{-\frac{t^2}{4s}}\, ds$$
$$= \frac{t\, e^{\gamma t}}{2\sqrt{2\pi}} \int_0^\infty p_{2/r}(x,y)\, r^{-1/2} e^{-2\gamma^2/r}\, e^{-\frac{t^2}{8} r}\, dr$$

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- From Biane, Pitman & Yor we know some Laplace transform

$$\int_0^\infty e^{-\frac{t^2}{8}s} f_{C_h}(s)\, ds = \left(\frac{1}{\cosh(t/2)}\right)^h, \quad h > 0 \qquad (1)$$
$$\int_0^\infty e^{-\frac{t^2}{8}s} f_{T_h}(s)\, ds = \left(\frac{\tanh(t/2)}{(t/2)}\right)^h, \quad h > 0 \qquad (2)$$

(C_h) and (T_h) two families of Lévy processes

Expression of p_t

- $$p_t(x,y) = \frac{\sqrt{\pi}W(y)}{2^{\alpha+\beta}}\frac{e^{\gamma^2 t}}{\sqrt{t}}$$
$$\sum_{n\geq 0}\frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n}P_n^{\alpha,\beta}(z)\left[\frac{(1+xy)}{8}\right]^n\left(f_{T_1}\star f_{C_{2n+\alpha+\beta+1}}\right)(\frac{2}{t}).$$

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- and ultraspherical case $\alpha=\beta$

$$p_t(x,y) = \sqrt{\pi}K_\alpha\frac{e^{\gamma^2 t}}{\sqrt{t}}W(y)$$
$$\sum_{n,k\geq 0}\frac{\Gamma(\nu(n,k,\alpha)+1)(xy)^k}{k!n!\Gamma(\alpha+n+1)}\left[\frac{(1-x^2)(1-y^2)}{4}\right]^n f_{T_1}\star f_{C_{\nu(n,k,\alpha)}}\left(\frac{1}{2t}\right)$$

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- $b < -1 \Rightarrow Y_t \in]-1, 1[$ p. s.
- From Girsanov formula, the generalized densities are given by

$$\left.\frac{dQ_a^b}{dQ_a^{b_0}}\right|_{\mathcal{F}_t}$$

$$= \exp\left\{(b - b_0)\int_0^t \frac{Y_s}{1 - Y_s^2}\, dY_s - \frac{1}{2}(b^2 - {b_0}^2)\int_0^t \frac{Y_s^2}{1 - Y_s^2}\, ds\right\}$$

- Maximum Likelihood Estimate of b :

$$\hat{b}_t = \frac{\displaystyle\int_0^t \frac{Y_s}{1 - Y_s^2}\, dY_s}{\displaystyle\int_0^t \frac{Y_s^2}{1 - Y_s^2}\, ds}$$

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$$S_{t,x} = \int_0^t \frac{Y_s}{1 - Y_s^2}\, dY_s - x \int_0^t \frac{Y_s^2}{1 - Y_s^2}\, ds$$

so that for $x > b$

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- $$\Lambda_{t,x}(\phi) = \frac{1}{t} \log E_b(e^{\phi S_{t,x}})$$

- From Itô formula,

$$F(Y_t) = -\frac{1}{2}\log(1 - Y_t^2) = \int_0^t \frac{Y_s}{1 - Y_s^2}\, dY_s + \frac{1}{2}\int_0^t \frac{1 + Y_s^2}{1 - Y_s^2}\, ds.$$

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$$\Lambda_{t,x}(\phi) = \frac{1}{t}\log E_{b(\phi,x)}(\exp(\{\phi+b-b(\phi,x))[F(Y_t)-F(y_0)-t/2]\})$$

$$b(\phi,x) = -1-\sqrt{(b+1)^2+2\phi(x+1)}$$

$$\Lambda_{t,x}(\phi)$$

$$= -\frac{\phi + b - b(\phi, x)}{2} + \frac{1}{t} \log \mathbb{E}_{b(\phi,x)}((1 - Y_t^2)^{-(\phi+b-b(\phi,x))/2})$$

$$= \Lambda_x(\phi) + \frac{1}{t} \log \frac{\sqrt{2\pi} K_{\alpha(\phi,x)} R_t(\phi, x)}{\sqrt{t}}$$

$$\Lambda_{t,x}(\phi) \longrightarrow_{t\to\infty} \Lambda_x(\phi)$$

Large deviations

Theorem: When $b \leq -1$, the family $\{\hat{b}_t\}_t$ satisfies a LDP with speed t and good rate function

$$J_b(x) = \begin{cases} -\frac{1}{4}\frac{(x-b)^2}{x+1} & \text{if} \quad x \leq x_0 \\ x + 2 + \sqrt{(b-x)^2 + 4(x+1)} & \text{if} \quad x > x_0 > b \end{cases}$$

where x_0 is the unique solution $x < -1$ of the equation

$$(b-x)^2 = 4x(x+1)\,.$$

Observation

Other cases:

- O-U case: (Florens & Pham, Bercu & Rouault):

$$J_1(x) = \begin{cases} -\frac{1}{4}\frac{(x-b)^2}{x} & \text{if} \quad x \le b/3 \\ 2x - b \quad \text{if} & x > x_0 > b/3 \end{cases}$$

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- squared Bessel case: (M.Z.):

$$I(x) = \begin{cases} \frac{(x-\nu)^2}{4x} & \text{if} \quad x \geq x_1 \\ 1 - x + \sqrt{(\nu - x)^2 - 4x} & \text{if} \quad x < x_1 \,. \end{cases}$$

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- squared O-U case: (M.Z.):

$$J_\delta(x) = \delta J_1$$

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Caracterisation

- Wong (1964) : principal solution of the Fokker-Planck eq. :

$$\partial_y^2[(B(y)p_t(x,y)] - \partial_y[(A(y)p_t(x,y)] = \partial_t(p_t(x,y)).$$

$\deg(B) = 2, \ \deg(A) = 1.$

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- Class of stationary Markov

$$\lim_{t\to\infty} p_t(x,y) = \int_{y_1}^{y_2} p_t(x,y)W(y)dy = W(x)\,,$$

W density function solution of the corresponding Pearson equation